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# Polynomial Sufficient Conditions of Well-Behavedness and Home Markings in Subclasses of Weighted Petri Nets
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Join-Free Petri nets, whose transitions have at most one input place, model systems without synchronizations while Choice-Free Petri nets, whose places have at most one output transition, model systems without conflicts. These classes respectively encompass the state machines (or S-systems) and the marked graphs (or T-systems).

Whereas a structurally bounded and structurally live Petri net is said to be "well-formed", a bounded and live Petri net is said to be "well-behaved". Necessary and sufficient conditions for the well-formedness of Join-Free and Choice-Free nets have been known for some time, yet the behavioral properties of these classes are still not well understood. In particular "polynomial" sufficient conditions for liveness, *i.e.* polynomial in time and with a polynomial initial number of tokens, have not been found until now. Besides, "home markings", which can be reached from every reachable marking thus allowing to construct systems that can return to their initial data distribution, are not well apprehended either for these subclasses.

We extend results on weighted T-systems to the class of weighted Petri nets and present transformations which preserve the language of the system and reduce the initial marking. We introduce a notion of "balancing" that makes possible the transformation of conservative systems into so-called "token-conservative" systems, whose number of tokens is invariant, while retaining the feasible transition sequences. This transformation is pertinent for all well-formed Petri nets and leads to polynomial sufficient conditions of liveness for well-formed Join-Free and Choice-Free nets. Finally, we also provide polynomial live and home markings for Fork-Attribution systems.

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## 1. INTRODUCTION

### 1.1. Models and analysis

Petri nets have proved useful to model discrete event systems possibly with conflicts, synchronization and concurrency. Whereas many of their properties are decidable [Murata 1989; Esparza 1998], their expressiveness, although not Turing-complete, comes at the cost of a high analysis complexity. The reachability and liveness problems, among others, are both EXPSPACE-hard while the $k$-boundedness problem is NP-hard [Jones et al. 1977] . In order to reduce this complexity, we focus on simpler classes that are expressive enough to model many real applications and simple enough to permit efficient analysis methods.

The formalism of Synchronous Data Flow (SDF) [Lee and Messerschmitt 1987] and its extension to Cyclo-Static Data Flow (CSDF) by adding phases [Engels et al. 1994] are special data flow models for concurrent applications to be executed on parallel architectures. They have been used in many—often multimedia—applications, such as MP3 playback [Wiggers et al. 2007]. Weighted T-systems [Teruel et al. 1992; Marchetti and Munier-Kordon 2009], also named generalized event graphs, are Petri nets having the same modeling power as the SDF, thus the methods developed for one model can be used for the other. However, as functionalities become more sophisticated, the expressiveness of models has to be extended. An objective is thus to generalize results common to SDF and T-systems in order to treat more complex applications involving choices.

Weighted S-systems can be viewed as dual of T-systems; while in a T-system a place cannot have more than one input and one output, the same is true for a transition in an S-system. Such systems model choices since their places may have several outputs, thus inducing a form of non-determinism.

Weighted Choice-Free systems augment the expressiveness of T-systems by allowing places to have several inputs. Weighted Join-Free systems expand the modeling power of S-systems as they do not constrain the number of transition outputs. The systems belonging to both of these larger classes, hence inheriting their structural properties, are called Fork-Attribution (FA) systems. Figure 1 represents the inclusion relations between the special classes of weighted Petri nets considered in this paper.

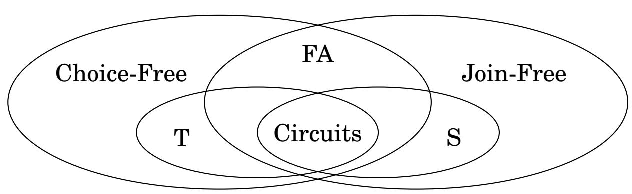



Fig. 1. Some classes and subclasses of weighted systems.

Embedded systems must be kept fully functionning over long periods of time. They have to preserve over time the entirety of their functions and use a limited amount of memory. Such guarantees are fundamental for applications. The corresponding notion in Petri nets is that of well-formedness. In Petri nets data is abstracted through the use of tokens, indicating the presence of data items while leaving out their values. Likewise, the values of a system's state are left out in the Petri net model, which uses only the bare bones of the state under the form of a vector of tokens, the marking.

Properties of Petri nets relate to the behavior of their markings. Well-formedness ensures that there exists at least one initialization of the net yielding a system whose functionalities are preserved—the liveness property—while a limited use of memory is ensured—the boundedness property. The structural notions of consistency and conservativeness are known necessary conditions for well-formedness [Sifakis 1978; Memmi and Roucairol 1980].

Well-formedness is characterized in polynomial time for several subclasses of weighted and ordinary (non-weighted) Petri nets. The well-formedness of ordinary Free-Choice nets, structurally more permissive than weighted Choice-Free nets but lacking weights, can be decided in polynomial time [Desel and Esparza 1995]. This is also the case for Equal-Conflict nets, which generalize weighted Choice-Free nets [Teruel and Silva 1996]. For several other weighted classes, there exist polynomial time necessary or sufficient conditions of well-formedness and sometimes complete characterizations [Recalde et al. 1995; 1996; Silva et al. 1998]. For the classes studied in this paper, namely weighted Choice-Free and Join-Free nets, well-formedness is decided in polynomial time [Recalde et al. 1996; Teruel et al. 1997; Amer-Yahia and Zerhouni 1999].

The next challenge is to find efficiently an initialization ensuring liveness and boundedness, a combination of properties called “well-behavedness”.

Polynomial time characterizations of well-behavedness are not known for weighted classes, even for weighted circuits, and may not exist. Thus, finding sufficient—although not necessary—conditions of well-behavedness that are polynomial in time and initial number of tokens would constitute a significant advance. In this paper, we shall simply qualify such conditions as “polynomial”.

However, even such relaxed conditions are not easy to discover. One has been found recently for T-systems [Marchetti and Munier-Kordon 2009]. In weighted Choice-Free systems and some larger classes, only costly conditions have been proposed [Recalde et al. 1996; Teruel et al. 1997], using exponential time algorithms or an exponential number of initial tokens, forbidding application to real systems. Necessary but not sufficient conditions for liveness and boundedness also exist [Amer-Yahia et al. 1999]. Other results encompass polynomial characterizations of liveness for several ordinary classes [Barkaoui and Minoux 1992; Chao and Nicdao 2001; Alimonti et al. 2011] and non-polynomial ones if weights are allowed with some restrictions [Barkaoui and Pradat-Peyre 1996; Jiao et al. 2004]. Well-behavedness is also not fully comprehended for weighted Join-Free systems.

Home markings are markings that can be reached from any reachable marking; they reflect a kind of cyclicity. Used as an initial data distribution in the system, they avoid a transient phase. This reversibility property is often required in embedded applications that need a steady behavior. Besides, such markings simplify the study of the reachability graph.

However, even in non-weighted Petri nets, home markings are not necessarily live [Murata 1989], witnessing to the difficulty of designing live and reversible systems. Under the liveness and boundedness hypothesis, weighted T-systems are known to be reversible [Teruel et al. 1992] while other weighted classes always have home markings [Teruel and Silva 1993; 1996; Recalde et al. 1998; Silva et al. 1998]. Their existence is stated, and even completely characterized for Choice-Free systems [Teruel et al. 1997], but their construction is not provided. Polynomial time algorithms that build live home markings with a polynomial number of initial tokens do not exist for most weighted classes, including Fork-Attribution systems.

In this paper, we show first that weighted conservative systems can be transformed, by modifying only the weights and the initial marking, into token-conservative systems, *i.e.* systems for which the total number of tokens is invariant. We prove next that this transformation preserves the language of the system and show that it can be used to simplify the study of the behavioral properties of any conservative system, including the liveness. As conservativeness is known to be induced by well-formedness, our transformation is general and does apply to all well-formed weighted Petri nets. We then focus on the weighted Choice-Free and Join-Free classes, for which we provide the first polynomial sufficient conditions of well-behavedness, whereas prior results used exponential methods. Finally, we construct the first polynomial live home marking for the intersection of these two classes, the Fork-Attribution class. For all the sufficient conditions above and this construction, the number of initial tokens can be bounded by a linear function of the weights. Thus, we extend the expressiveness of the models usable in real applications, whose well-behavedness and reversibility can be ensured efficiently.

Comparing with [Delosme et al. 2013], this paper provides more details about the transformations, improves upon the polynomial live marking for well-formed Choice-Free systems and introduces the first polynomial live and home marking in Fork-Attribution systems.

### 1.2. Models and applications

Join-Free, Choice-Free and S-systems are appealing not only from a theoretical point of view but also because they can model a variety of useful applications.

*1.2.1. S-systems.* They are a simple subclass of weighted Petri nets, in which programs read or write in a single memory, while memories can be shared between programs. They allow to model asynchronous parallel algorithms on several processors. Asynchronous methods cut down the number of synchronization points between processors, eliminating idle time at the expense of extra computations, yet they may perform better than their synchronous counterparts. Computational models as well as an associated convergence theory have been developed [Frommer and Szyld 2000]. A powerful and simple model reads data from a shared memory, computes a function and overwrites data in the common memory with the corresponding updated values. This computational model applies to a wide range of problems, including solving nonsingular linear systems, and is represented by a Petri net in Figure 2.

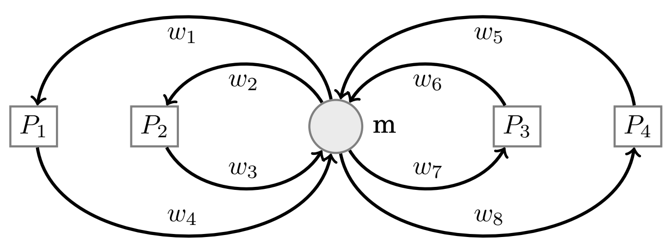



Fig. 2. The S-net models one shared memory as the place $m$, in which four asynchronous parallel processors read and write to compute an asynchronous iteration. Weights denote amounts of read or written data.

*1.2.2. Join-Free systems.* They are closely related to Basic Parallel Processes (BPP, context-free multiset rewrite systems, commutative context-free processes) [Mayr 2000]. Results on Join-Free systems with a weight restriction, named communication-free, have served to gain insight into commutative context-free grammars and BPP [Esparza 1997].

*1.2.3. Choice-Free systems.* Flow applications are usually modeled with weighted T-systems (equivalently with SDFs) [Lee and Messerschmitt 1987]. The weighted Choice-Free model adds the possibility to write asynchronously in a memory, thus is strictly more powerful than the T-system model.

### 1.3. Organization of the paper

In Section 2, we recall classical definitions, notations and properties. In Section 3, we define the *scaling* and *balancing* of systems together with the trimming down to *useful tokens*, which are polynomial time transformations that preserve the set of feasible firing sequences. Moreover, balancing is compared to the notion of *normalization* that was developed for T-systems. In Section 4, we use balancing to present a polynomial sufficient condition of liveness for well-formed balanced Join-Free nets. We then use this condition to deduce a polynomial live marking for well-formed Join-Free nets that are not necessarily balanced. In Section 5, we exploit the marking developed for well-formed Join-Free nets to construct a polynomial live marking for well-formed Choice-Free nets. In Section 6, we show that neither one of these sufficient conditions is necessary in the weighted case. Finally, in Section 7, we provide a polynomial live home marking for Fork-Attribution systems.

## 2. DEFINITIONS, NOTATIONS AND PROPERTIES

We recall in this section some basic definitions and results concerning P/T nets, and systems. After an overview of notations for weighted nets, we recall the definitions of special classes of nets considered in our study, namely Choice-Free, Join-Free and some of their subclasses. We then provide some definitions relevant to markings and firing sequences, and their relationships. Finally we review some definitions and results related to liveness and boundedness.

### 2.1. Weighted and ordinary nets

A *(weighted) net* is a triple $N = (P, T, W)$ where:

– the sets $P$ and $T$ are finite and disjoint, $T$ contains only transitions and $P$ only places,
– $W : (P \times T) \cup (T \times P) \mapsto \mathbb{N}$ is a non-negative function.

$P \cup T$ is the set of the elements of the net.

An arc is present from a place $p$ to a transition $t$ (resp. a transition $t$ to a place $p$) if $W(p, t) > 0$ (resp. $W(t, p) > 0$). An *ordinary* net is a weighted net whose weighting function $W$ takes values in $\{0, 1\}$.

The *incidence matrix* of a net $N = (P, T, W)$ is a place-transition matrix $C$ defined as

$$\forall p \in P \;\; \forall t \in T, \;\; C[p, t] = W(t, p) - W(p, t)$$

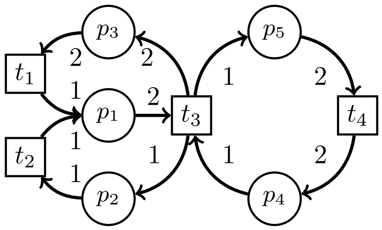
where the weight of any non-existing arc is $0$.

The *pre-set* of the element $x$ of $P \cup T$ is the set $\{w | W(w, x) > 0\}$, denoted by ${}^\bullet x$. By extension, for any subset $E$ of $P$ or $T$, ${}^\bullet E = \bigcup_{x \in E} {}^\bullet x$.

The *post-set* of the element $x$ of $P \cup T$ is the set $\{y | W(x, y) > 0\}$, denoted by $x^\bullet$. Similarly, $E^\bullet = \bigcup_{x \in E} x^\bullet$.

We denote by $max_p^N$ the maximum output weight of $p$ in the net $N$ and by $gcd_p^N$ the greatest common divisor of all input and output weights of $p$ in the net $N$. The simpler notations $max_p$ and $gcd_p$ are used when no confusion is possible. We denote by $\mathbb{1}^n$ the column vector of size $n$ whose components are all equal to $1$.

Figure 3 presents a weighted net and its corresponding incidence matrix.




| | $t_1$ | $t_2$ | $t_3$ | $t_4$ |
|---|---|---|---|---|
| $p_1$ | 1 | 1 | -2 | 0 |
| $p_2$ | 0 | -1 | 1 | 0 |
| $p_3$ | -2 | 0 | 2 | 0 |
| $p_4$ | 0 | 0 | -1 | 2 |
| $p_5$ | 0 | 0 | 1 | -2 |

Fig. 3. A weighted net and the corresponding incidence matrix.

A P-subnet $S = (P', T', W')$ of a net $N = (P, T, W)$ is generated by a subset of places $P' \subseteq P$ and is such that $T' = {}^\bullet P' \cup P'^\bullet$. $W'$ is the restriction of $W$ to $P'$ and $T'$.

Figure 4 pictures two P-subnets of the net from Figure 3.


Fig. 4. Two P-subnets of the net pictured by Fig. 3, defined by the sets of places $\{p_1, p_2, p_3\}$ and $\{p_4, p_5\}$.

### 2.2. Special classes of nets

$N = (P, T, W)$ is a (weighted) *Choice-Free net* if any place has at most one output transition, *i.e.* $\forall p \in P$, $|p^\bullet| \leq 1$. A ***T-net*** is a Choice-Free net such that any place has at most one input transition, *i.e.* $\forall p \in P$, $|{}^\bullet p| \leq 1$.

$N = (P, T, W)$ is a (weighted) *Join-Free net* if any transition has at most one input place, *i.e.* $\forall t \in T$, $|{}^\bullet t| \leq 1$. An *S-net* is a Join-Free net such that any transition has at most one output place, *i.e.* $\forall t \in T$, $|t^\bullet| \leq 1$.

A *Fork-Attribution* net (or FA net) is both a Join-Free and a Choice-Free net.

Since a transition of a T-net may have several input places, T-nets are not included in FA nets. Similarly, since a place of an S-net may have several output transitions, S-nets are not included in FA nets. The nets presented in Figure 3 and Figure 4 are Choice-Free. The net on the left side of Figure 4 is an FA net while the net on the right side is a circuit, hence a particular T-net, S-net and FA net.

The *dual* of a net is defined by reversing the arcs and swapping places and transitions. This transformation amounts to transposing the incidence matrix.

Choice-Free and Join-Free classes are dual. S and T classes are also dual. However transforming a net into its dual does not necessarily provide a simple way to deduce behavioral properties of one net from the other.

### 2.3. Markings and firing sequences

A *marking* $M$ of a net $N$ is a mapping $M : P \to \mathbb{N}$. We shall also denote by $M$ the column vector whose components are the values $M(p)$ for $p \in P$. A marking $M$ that is restricted to a subset of places $P'$ is denoted by $M[P']$.

A *system* is a couple $(N, M_0)$ where $N$ is a net and $M_0$ the initial marking of $N$.

A marking $M$ of a net $N$ *enables* a transition $t \in T$ if $\forall p \in {}^\bullet t, M(p) \geq W(p,t)$. A marking $M$ *enables* a place $p \in P$ if $\forall t \in p^\bullet, M(p) \geq W(p,t)$. The marking $M'$ obtained from $M$ by the firing of an enabled transition $t$, noted $M \xrightarrow{t} M'$, is defined by $\forall p \in P, M'(p) = M(p) - W(p,t) + W(t,p)$.

A *firing sequence* $\sigma$ of length $n \geq 1$ on the set of transitions $T$ is a mapping $\{1, \ldots, n\} \to T$. A sequence is *infinite* if its domain is countably infinite. A firing sequence $\sigma = t_1 t_2 \cdots t_n$ is *feasible* if the successive markings obtained, $M_0 \xrightarrow{t_1} M_1 \xrightarrow{t_2} M_2 \cdots \xrightarrow{t_n} M_n$, are such that $M_{i-1}$ enables transition $t_i$ for any $i \in \{1, \cdots, n\}$. We note $M_0 \xrightarrow{\sigma} M_n$.

The *Parikh vector* $\vec{\sigma} : T \to \mathbb{N}$ associated with a finite sequence of transitions $\sigma$ maps every transition $t$ of $T$ to the number of occurrences of $t$ in $\sigma$.

A marking $M'$ is said to be *reachable* from the marking $M$ if there exists a feasible firing sequence $\sigma$ such that $M \xrightarrow{\sigma} M'$. The set of reachable markings from $M$ is denoted by $[M\rangle$.

A *home marking* is a marking that can be reached from any reachable marking. Formally, $M$ is a home marking in the system $(N, M_0)$ if $\forall M' \in [M_0\rangle, M \in [M'\rangle$. A system is *reversible* if its initial marking is a home marking.

### 2.4. Liveness and boundedness

Liveness and boundedness are two basic properties ensuring that all transitions of a system $S = (N, M_0)$ can always be fired and that the overall number of tokens remains bounded. More formally,

– A system $S$ is *live* if for every marking $M$ in $[M_0\rangle$ and for every transition $t$, there exists a marking $M'$ in $[M\rangle$ enabling $t$.
– $S$ is *bounded* if there exists an integer $k$ such that the number of tokens in each place never exceeds $k$. Formally,

$$\exists k \in \mathbb{N} \;\; \forall M \in [M_0\rangle \;\; \forall p \in P, \;\; M(p) \leq k\,.$$

  $S$ is $k$*-bounded* if, for any place $p \in P$,

$$k \geq \max\{M(p) | M \in [M_0\rangle\}\,.$$

– A system $S$ is *well-behaved* if it is live and bounded.

A marking $M$ is live (resp. bounded) for a net $N$ if the system $(N, M)$ is live (resp. bounded).

The structure of a net $N$ may be studied to ensure the existence of an initial marking $M_0$ such that $(N, M_0)$ is live and bounded:

– A net $N$ is *structurally live* if there exists a marking $M_0$ such that $(N, M_0)$ is live.
– A net $N$ is *structurally bounded* if the system $(N, M_0)$ is bounded for every $M_0$.
– A net is *well-formed* if it is structurally live and structurally bounded.

Semiflows are particular left or right annulers of the incidence matrix $C$ of a net:

– A P-semiflow is a non-null vector $X \in \mathbb{N}^{|P|}$ such that ${}^tX \cdot C = 0$.
– A T-semiflow is a non-null vector $Y \in \mathbb{N}^{|T|}$ such that $C \cdot Y = 0$.

A P-semiflow is *minimal* if the greatest common divisor of its components is equal to $1$ and its support is not a proper superset of the support of any other P-semiflow. The same definition applies to T-semiflows.

Our study focuses on well-formed nets. For Choice-Free and Join-Free nets, well-formedness is related to the conservativeness and the consistency properties defined as follows using the incidence matrix $C$ of a net $N$:

– $N$ is *conservative* if there exists a P-semiflow $X \in \mathbb{N}^{|P|}$ for $C$ such that $X \geq \mathbb{1}^{|P|}$.
– $N$ is *consistent* if there exists a T-semiflow $Y \in \mathbb{N}^{|T|}$ for $C$ such that $Y \geq \mathbb{1}^{|T|}$.

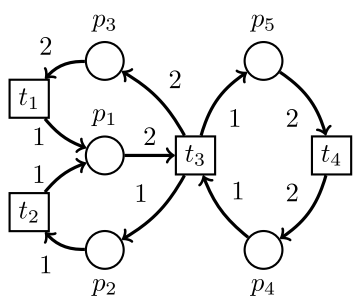
The next theorem expresses a necessary and sufficient condition of well-formedness for Choice-Free and Join-Free nets.

THEOREM 2.1 ([RECALDE ET AL. 1996; TERUEL ET AL. 1997]). *Suppose that $N$ is a weighted and strongly connected Choice-Free or Join-Free net. The properties*

– *$N$ is consistent and conservative*
– *$N$ is well-formed*

*are equivalent. Moreover, consistency implies conservativeness in strongly connected Choice-Free nets and conservativeness implies consistency in strongly connected Join-Free nets.*

Figure 5 shows a well-formed Choice-Free net.




$$\begin{pmatrix} 1 & 1 & -2 & 0 \\ 0 & -1 & 1 & 0 \\ -2 & 0 & 2 & 0 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 2 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & -2 & 0 \\ 0 & -1 & 1 & 0 \\ -2 & 0 & 2 & 0 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 1 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix}$$

Fig. 5. A strongly connected weighted Choice-Free net that is consistent (the right vector ${}^t(2,2,2,1)$ is a T-semiflow and its components are $\geq 1$) hence conservative (the left vector $(2,2,1,1,1)$ is a P-semiflow and its components are $\geq 1$), thus well-formed.

## 3. TRANSFORMATIONS PRESERVING FIRING SEQUENCES

This section presents several polynomial transformations that preserve the feasible firing sequences. They will be used in the sequel to prove the sufficient conditions of liveness. We first define the *scaling* of a system by a vector. The *balancing* transformation is introduced next that can be applied to a system when it is conservative to obtain what we call a *token-conservative* system, whose total number of tokens is invariant. *Normalization* of T-systems is then recalled and compared to balancing. We finally present the *useful tokens* property allowing to reduce the initial number of tokens without modifying the feasible sequences.

### 3.1. Scaling of systems

We define the *scaling*, multiplying weights and initial markings by positive rational numbers. We then show that the obtained system has the same language as the initial system and study the repercussions of the transformation on reachable markings as well as T-semiflows.

*Definition* 3.1. The multiplication of all input and output weights of a place $p$ together with its marking by a positive rational $\alpha_p$ is a *scaling of the place* $p$ if the resulting input and output weights and marking are integers. If each place $p$ of a system is scaled by a positive rational $\alpha_p$, the *system* is said to be *scaled by the vector* $\alpha$ whose components are the scaling factors $\alpha_p$.

Figure 6 shows the scaling of a marked place by $2$.


Fig. 6. The marked place on the left is scaled by $2$, yielding the place on the right.

THEOREM 3.2. *Let $S = ((P, T, W), M_0)$ be a system and $\alpha$ a vector of $|P|$ positive rational components. Scaling $S$ by $\alpha$ preserves the feasible sequences of firings.*

PROOF. Let $S' = ((P, T, W'), M_0')$ be the system obtained when scaling the system $S = ((P, T, W), M_0)$ by $\alpha$. Let $\sigma = \sigma_1 t$ be a finite sequence of firings. We prove by induction on the size of $\sigma$ that $\sigma$ is equivalently feasible in $S$ and $S'$.

The property is true if $\sigma$ is empty. Now suppose that $\sigma_1$ is feasible in both $S$ and $S'$. The firing sequence $\sigma$ is feasible in $S$ if and only if

$$\forall p \in {}^\bullet t,\ M_0(p) + \sum_{t_i \in {}^\bullet p} W(t_i, p) \cdot \vec{\sigma}_1(t_i) - \sum_{t_i \in p^\bullet} W(p, t_i) \cdot \vec{\sigma}_1(t_i) \geq W(p, t)$$

which gives after scaling

$$\forall p \in {}^\bullet t,\ \alpha_p \cdot M_0(p) + \sum_{t_i \in {}^\bullet p} \alpha_p \cdot W(t_i, p) \cdot \vec{\sigma}_1(t_i) - \sum_{t_i \in p^\bullet} \alpha_p \cdot W(p, t_i) \cdot \vec{\sigma}_1(t_i) \geq \alpha_p \cdot W(p, t)$$

and is equivalent to

$$\forall p \in {}^\bullet t,\ M_0'(p) + \sum_{t_i \in {}^\bullet p} W'(t_i, p) \cdot \vec{\sigma}_1(t_i) - \sum_{t_i \in p^\bullet} W'(p, t_i) \cdot \vec{\sigma}_1(t_i) \geq W'(p, t)\,.$$

Thus, $t$ is equivalently enabled in $S'$. □

As shown in the previous theorem, the operation of scaling does not change the language of the system. We deduce that the liveness property is consequently preserved as stated by the following corollary.

COROLLARY 3.3. *A system is live if and only if its scalings are live.*

THEOREM 3.4. *Scaling preserves the set of T-semiflows.*

PROOF. Consider a system with incidence matrix $C$ and a scaling vector $\alpha$. When the system is scaled by $\alpha$, the incidence matrix $C'$ is obtained which satisfies $C'[p] = \alpha_p \cdot C[p]$ for every place $p$. These multiplications of rows by *non-zero* coefficients $\alpha_p$ preserve the kernel (or null space) of the matrix. □

The following result completes the study of the scaling transformation. It shows that the system obtained by firing a feasible sequence $\sigma$ and then scaling it by a vector $\alpha$, is the same as the system obtained by first scaling the initial system by $\alpha$ and then firing $\sigma$. Thus, firing and scaling commute (see Figure 7).

THEOREM 3.5. *Consider a system $S$ with initial marking $M_0$ and a feasible sequence $\sigma$. Call $M$ the reachable marking such that $M_0 \xrightarrow{\sigma} M$. Let $\alpha$ be a scaling vector for $S$ and denote $S'$ the associated scaled system with initial marking $M'_0 = \alpha \circ M_0$, where $\circ$ is the component-wise product (Hadamard product). The reachable marking $M'$ in $S'$ such that $M'_0 \xrightarrow{\sigma} M'$ satisfies $M' = \alpha \circ M$.*

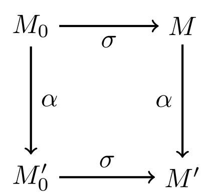



Fig. 7. Scaling and firing commute: $M'_0 = \alpha \circ M_0$ and $M' = \alpha \circ M$.

PROOF. The vector $\alpha$ scales $S = ((P, T, W), M_0)$ to $S' = ((P, T, W'), M'_0)$ while preserving the sequences of firings (Theorem 3.2). Let $\sigma$ be a sequence feasible in both $S$ and $S'$, then for every place $p$ and rational scaling coefficient $\alpha_p > 0$,

$$M'(p) = M'_0(p) + \sum_{t_i \in {}^\bullet p} W'(t_i, p) \cdot \vec{\sigma}(t_i) - \sum_{t_i \in p^\bullet} W'(p, t_i) \cdot \vec{\sigma}(t_i)$$

$$= \alpha_p \cdot \Big( M_0(p) + \sum_{t_i \in {}^\bullet p} W(t_i, p) \cdot \vec{\sigma}(t_i) - \sum_{t_i \in p^\bullet} W(p, t_i) \cdot \vec{\sigma}(t_i) \Big) .$$

Since

$$M(p) = M_0(p) + \sum_{t_i \in {}^\bullet p} W(t_i, p) \cdot \vec{\sigma}(t_i) - \sum_{t_i \in p^\bullet} W(p, t_i) \cdot \vec{\sigma}(t_i) \, ,$$

it follows that $M'(p) = \alpha_p \cdot M(p)$. □

### 3.2. Balancing and token-conservation in weighted Petri Nets

When transition firings preserve the total number of tokens in a system, the study of its behavioral properties is greatly simplified. This property, which is a special case of conservativeness, has been introduced under the name $1$-*conservativeness* in [Jones et al. 1977]. It is also sometimes called strict conservativeness, and, in this paper, we shall call it *token-conservativeness* or, even better, *token-conservation*, since its meaning is then readily understood. In the following, balancing is defined as a scaling that yields token-conservative systems and applies to all conservative systems.

*Definition* 3.6. A transition $t$ is token-conservative if

$$\sum_{p \in {}^\bullet t} W(p,t) = \sum_{p \in t^\bullet} W(t,p)\,.$$

If all the transitions of a net are token-conservative, the net is said to be token-conservative.

Now we define *balancing*, which transforms a system into a token-conservative system having the same set of feasible firing sequences.

*Definition* 3.7. *Balancing* a system $S$ consists in scaling $S$ by a vector of positive rational numbers such that the resulting system is token-conservative.

This transformation can help gain insight into conservative systems as shown by the next theorem.

THEOREM 3.8. *A system is conservative if and only if it can be balanced.*

PROOF. Consider a conservative system with incidence matrix $C$, then by definition there exists a vector $X \geq \mathbb{1}^{|P|}$ of natural numbers such that ${}^tX \cdot C = 0$. Multiplying every component $C[p,t]$ by $X_p$ yields an incidence matrix $C'$ satisfying for every transition $t$, ${}^t\mathbb{1}^{|P|} \cdot C'[t] = 0$. The new initial marking contains only integers and the new system is balanced.

Conversely, if a system can be balanced, there exists a vector $X$ with positive rational numbers that annuls every column of its incidence matrix. The multiplication of $X$ by the least common multiple of the denominators of its components gives a conservative vector, hence the system is conservative. □

COROLLARY 3.9. *A conservative system is live if and only if its balancings are live.*

PROOF. Corollary 3.3 applies as balancing is a particular scaling. □

Conservativeness is a known necessary condition for the well-formedness of weighted Petri nets [Sifakis 1978; Memmi and Roucairol 1980]. Finding an adequate (conservative) balancing vector for a conservative net with incidence matrix $C$ consists in finding a solution to ${}^tX \cdot C = 0, X \geq \mathbb{1}^{|P|}$, or ${}^tC \cdot X = 0, X \geq \mathbb{1}^{|P|}$ where the entries of $C$ are integers and those of $X$ are natural numbers.

As the systems we study are well-formed, they are conservative. Thus, a positive rational solution $X$ can be found with a linear program in weakly polynomial time [Megiddo 1987]. Multiplying the components of $X$ by the product of their denominators leads to a solution whose components are positive natural numbers with a polynomial increase of the number of bits. However, the obtained values and thus the number of tokens can grow exponentially, inducing an exponential increase of the buffer sizes.

In the next sections, we construct live and home markings in the balanced version of the system, then we use the language preservation property to construct initial markings in the original system with a polynomial number of initial tokens. Thus, balancing is only used in the proofs, leading to polynomial sufficient conditions.

### 3.3. Normalization and its relation to balancing

Normalization was introduced in the context of weighted T-systems to obtain a sufficient condition of liveness [Marchetti and Munier-Kordon 2009]. By applying this transformation, the number of tokens remains constant in every circuit of the T-system, simplifying the study of its behavior. Normalization is explicited in a different way here starting from the notion of consistency of a well-formed T-system. It is finally compared to balancing in S-systems.

*Definition* 3.10. A transition $t$ is *normalized* if all the input and output weights of $t$ are equal. A net, or a system, is normalized if all its transitions are normalized.

Normalizing transforms a system into a normalized one by means of an appropriate system scaling. The next theorem shows that normalization can be performed on any consistent weighted T-system.

THEOREM 3.11 ([MARCHETTI AND MUNIER-KORDON 2009]). *Weighted T-systems can be normalized if they are consistent.*

PROOF. Suppose that $S = (N, M_0)$ is a weighted consistent T-system with incidence matrix $C$. There exists by definition a vector $Y \geq \mathbb{1}^{|T|}$ of integers such that $C \cdot Y = 0$.

Since S is consistent, its places either have one input and one output transition or are isolated. Only non isolated places need be considered for normalization.

Denote by $K$ the least common multiple of the values $Y[t]$, $t \in T$. We observe that, since any place $p$ has exactly one input transition $t'$ and one output transition $t$,

$$W(t', p) \cdot Y[t'] = W(p, t) \cdot Y[t]$$

and we set

$$\alpha_p = \frac{K}{W(t', p) \cdot Y[t']} = \frac{K}{W(p, t) \cdot Y[t]} .$$

Now we prove that the weighted T-system $S' = (N', M'_0)$ obtained by scaling $S$ with $(\alpha_1, \cdots, \alpha_{|P|})$ is normalized.

Consider a transition $t$, then for every place $p \in {}^\bullet t$

$$W'(p, t) = \alpha_p \cdot W(p, t) = \frac{K}{Y[t]} .$$

Similarly, for any place $p \in t^\bullet$,

$$W'(t, p) = \alpha_p \cdot W(t, p) = \frac{W(t, p) \cdot K}{W(t, p) \cdot Y[t]} = \frac{K}{Y[t]} .$$

Thus every transition $t$ is normalized and $S'$ is normalized. □

Well-formed strongly connected T-systems are consistent and conservative (Theorem 2.1), thus they can be either normalized or balanced. When a system is normalized, the number of tokens is invariant in every circuit while, when it is balanced, its total number of tokens is kept constant. Figure 8 shows the difference between a normalized T-system and a balanced T-system.

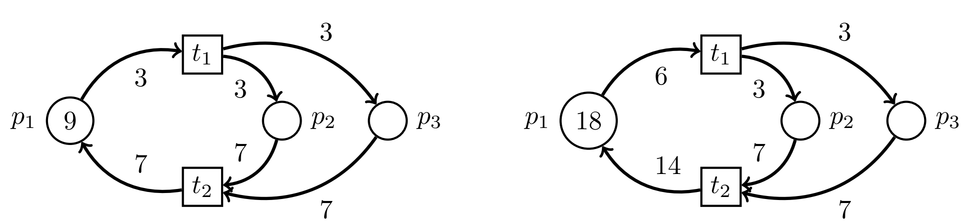



Fig. 8. Consistent T-systems can be normalized. Since they are also conservative they can be balanced as well. The T-system on the left is normalized but not balanced and all circuits preserve their number of tokens when firing, take e.g. the circuit $p_1 t_1 p_2 t_2$. The equivalent T-system on the right is obtained by multiplying $p_1$ by 2: it is balanced but not normalized and the total number of tokens remains constant after any sequence of firings.

S-systems constitute a subclass of Join-Free systems. As shown below, the normalization that was developed for T-systems coincides with balancing for S-systems.

THEOREM 3.12. *A strongly connected S-system is balanced if and only if it is normalized.*

PROOF. As each of its transitions has just one input and one output place, the system is balanced if and only if each transition's unique input weight equals its unique output weight, characterizing a normalized system. □

However, the results of balancing and normalization differ in other weighted classes, even if the considered systems are well-formed, consistent and conservative. Figure 9 depicts a normalized and balanced S-system on the left, and, on the right, a Fork-Attribution system, that is balanced but neither normalized nor even normalizable.

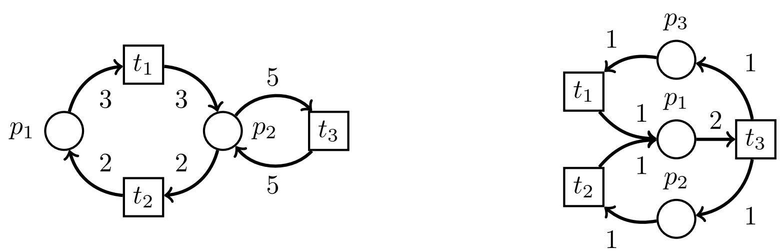



Fig. 9. Capacity to normalize or balance a system depends on its structure. Conservative systems can be balanced, hence all well-formed systems can be balanced. A well-formed balanced FA system may not be normalized nor normalizable.

### 3.4. Useful tokens in weighted Petri nets

We show that some tokens do not impact the behavior of a system and hence can be removed without modifying the language of the system.

*Definition* 3.13. A weighted Petri net is said to satisfy the *useful tokens condition* if every place $p$ is initially marked with a multiple of $gcd_p$.

The following theorem shows that any initial marking can be modified to satisfy this condition in such a way that the set of feasible sequences is not modified.

THEOREM 3.14. *The marking $M_0(p)$ of every place $p$ of a system $S = (N, M_0)$ can be replaced by*

$$\left\lfloor \frac{M_0(p)}{gcd_p} \right\rfloor \cdot gcd_p$$

*without modifying the feasible firing sequences of $S$.*

PROOF. From the initial marking $M_0$ of the net $N$ we construct a marking $M_0'$ according to

$$\forall p \in P,\ M_0'(p) = \left\lfloor \frac{M_0(p)}{gcd_p} \right\rfloor \cdot gcd_p\,.$$

Let $r_p$ be the remainder of the division of $M_0(p)$ by $gcd_p$, hence $M_0(p) = M_0'(p) + r_p$. We prove by induction on the size of a firing sequence $\sigma$ that $\sigma$ is feasible for $M_0$ if and only if $\sigma$ is feasible for $M_0'$.

The property is true if $\sigma$ is empty. Otherwise, suppose $\sigma_1$ is feasible in both systems, with $M_0 \xrightarrow{\sigma_1} M$ and $M'_0 \xrightarrow{\sigma_1} M'$, and consider $\sigma = \sigma_1 t$. Denote

$$v_p = \sum_{t_i \in {}^\bullet p} W(t_i, p) \cdot \vec{\sigma}_1(t_i) - \sum_{t_i \in p^\bullet} W(p, t_i) \cdot \vec{\sigma}_1(t_i)\,.$$

Since all input and output weights of $p$ are multiples of $gcd_p$, $v_p$ is a multiple of $gcd_p$. By definition of $M_0$, we have the following equivalence on the possibility to fire $t$ in both systems after the firing of $\sigma_1$:

$$\forall p \in {}^\bullet t, M_0(p) + v_p \geq W(p,t) \Longleftrightarrow \forall p \in {}^\bullet t, M'_0(p) + r_p + v_p \geq W(p,t)\,.$$

Since $M'_0(p)$, $v_p$ and $W(p,t)$ are multiples of $gcd_p$ for every place $p$, while $r_p$ is strictly smaller than $gcd_p$, we deduce

$$\forall p \in {}^\bullet t, M'_0(p) + v_p \geq W(p,t)\,.$$

Therefore $t$ can equivalently be fired at $M'$, which completes the proof. □

## 4. WELL-BEHAVEDNESS OF JOIN-FREE SYSTEMS

We present in this section a polynomial sufficient condition for the liveness of balanced Join-Free systems and a polynomial live marking for well-formed Join-Free systems. The latter will be used in Section 5 to deduce a polynomial live marking for well-formed Choice-Free systems.

Observing that a well-formed Join-Free system is necessarily balanceable, we express a first sufficient condition of liveness on a balanced version of the system. Then we examine the special case of ordinary well-formed Join-Free systems for which we derive a simple necessary and sufficient condition of liveness. Finally, we propose a live marking that imposes a distribution of the initial tokens among all places and may be used in all balanceable Join-Free systems.

### 4.1. Well-formedness of Join-Free systems and balancing

We consider in this section only well-formed Join-Free systems. By Theorem 2.1, such systems are conservative. According to Theorem 3.8 these systems may also be balanced, which can be done in polynomial time. Hence we shall restrict our study to Join-Free balanceable systems.

### 4.2. A sufficient polynomial condition for the liveness of balanced Join-Free systems

The following technical lemma expresses a simple sufficient condition for the existence of enabled places.

LEMMA 4.1. *Let $S = ((P,T,W), M_0)$ be a balanced strongly connected Join-Free system satisfying the useful tokens condition and the inequality*

$$\sum_{p \in P} M_0(p) > \sum_{p \in P} (max_p - gcd_p)\,. \tag{1}$$

*Then for every marking $M$ in $[M_0\rangle$, there exists a place $p \in P$ which is enabled by $M$.*

PROOF. As $M_0$ fulfills the useful tokens condition, it follows that for every place $p$, $M_0(p)$ is a multiple of $gcd_p$. Since the input and output weights of every place $p$ are also multiples of $gcd_p$, $M(p)$ is a multiple of $gcd_p$ for every reachable marking $M$ and every place $p$.

Now suppose, by contradiction, that $M$ is a fixed reachable marking that does not enable any place. Then

$$\forall p \in P, M(p) \leq max_p - gcd_p,$$

hence

$$\sum_{p \in P} M(p) \leq \sum_{p \in P} (max_p - gcd_p)\,. \tag{2}$$

Since $S$ is balanced, every transition firing maintains the number of tokens in the system, implying that

$$\sum_{p \in P} M(p) = \sum_{p \in P} M_0(p)\,.$$

Inequality (2) is therefore equivalent to

$$\sum_{p \in P} M_0(p) \leq \sum_{p \in P} (max_p - gcd_p),$$

contradicting inequality (1). □

We are now able to prove that inequality (1) provides a sufficient condition for liveness.

THEOREM 4.2. *A balanced strongly connected Join-Free system $S = (N, M_0)$ satisfying the useful tokens condition is live if*

$$\sum_{p \in P} M_0(p) > \sum_{p \in P} (max_p - gcd_p)\,.$$

PROOF. Let $S$ be a Join-Free system meeting the conditions of the theorem. We show that $S$ is live, that is, for every reachable marking $M$ and every transition $t$, there exists a finite and feasible firing sequence starting at $M$ that leads to a marking $M'$ enabling $t$. For that purpose, we prove that Algorithm 1 computes such a sequence and terminates. Tokens are arbitrarily numbered to ensure its convergence.

If $M(p) \geq W(p,t)$ then $t$ is enabled. The algorithm terminates and $\sigma$ is the requested firing sequence.

Consider now the alternative. The place $p$ is not enabled. $L$ is not empty by Lemma 4.1 and thus, $p'$ exists. Moreover $p \neq p'$ since $p \notin L$.

At every step of the loop, a firing occurs so as to reduce the minimal distance between the mobile token with smallest number and the place $p$. Note that a firing can move several tokens at once on different paths, taking some of them away from $p$. Such a firing is always possible when $p$ is not enabled, since by Lemma 4.1 there exists at least one enabled place at any reachable marking. In so doing, a new marking $M'$ is reached, inducing the new shortest distances of $d^{M'}$. We prove that $d^M >_{lex} d^{M'}$, following the lexical order on $\mathbb{N}^\delta$. The firing of $t'$ ensures that $d_i^{M'} < d_i^M$. At this step, as the numbers of the other displaced tokens are greater than $i$, the inequality $d^M >_{lex} d^{M'}$ is true, even though $d_j^M$ may increase for $j > i$. Now, as the lexical order is well-founded over $\mathbb{N}^\delta$, the algorithm terminates, $\sigma$ is finite and $t$ is enabled. □

Figure 10 shows the application of this theorem to a balanced Join-Free system. The pictured system satisfies $\sum_p (max_p - gcd_p) = 1 + 1 + 0 = 2$. The inequality becomes $\sum_p M_0(p) > 2$ and is satisfied by the marking on the figure.

### 4.3. The special case of ordinary Join-Free systems

We show that the sufficient condition of liveness for well-formed Join-Free systems becomes a necessary and sufficient condition of liveness for ordinary well-formed Join-Free systems, which are S-systems.

**ALGORITHM 1:** The algorithm computes a feasible firing sequence starting at $M$ so as to enable any fixed transition $t$.

**Data**:
- The current reached marking $M$, which contains $\delta$ tokens arbitrarily numbered $1, \dots, \delta$;
- The unique input place $p$ of $t$;
- The $\delta$-tuple $d^M = (d_1^M, \dots, d_\delta^M)$, which associates the shortest distance $d_i^M$ from token $i$ to the place $p$ according to the marking $M$;

**Result**: A finite firing sequence $\sigma$ such that $M \xrightarrow{\sigma} M'$ and $M'$ enables $t$.

$\sigma := \epsilon$, the empty sequence;

**while** $M(p) < W(p, t)$ **do**

 Let $L$ be the set of the places enabled by $M$ and $J = \{i_1 \dots i_k\}$ the set of the numbers of the tokens in the places of $L$ at $M$;

 Let $p'$ be the place of $L$ containing token $i$ where $i$ is the smallest value in $J$;

 Let $\mu = p', t', p'', \dots, p$ be a shortest path from $p'$ to $p$;

 Fire $t'$ such that token $i$ is sent to $p''$;

 Update $M$ and $d^M$;

 $\sigma := \sigma\ t'$

**end**

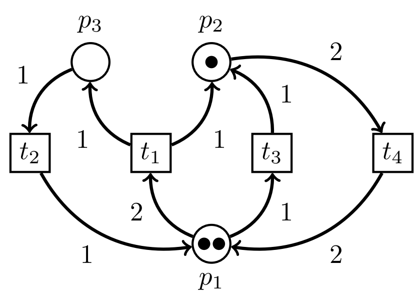



Fig. 10. The initial marking of this balanced Join-Free system fulfills the conditions of Theorem 4.2 and is thus well-behaved.

THEOREM 4.3. *An ordinary strongly connected well-formed Join-Free net is an S-net.*

PROOF. By contradiction, suppose that there exists an ordinary well-formed strongly connected Join-Free net $N$ and a transition $t$ with $|t^\bullet| > 1$. $N$ is strongly connected, thus there exists a circuit $c = tp_1t_1p_2t_2 \dots p_kt_k$ passing through $t$ with $t_k = t$. As $N$ is conservative by Theorem 2.1, there exists $X \geq \mathbb{1}^{|P|}$ such that ${}^tX \cdot C = 0$, where $C$ is the incidence matrix of $N$. Since by assumption $|t^\bullet| > 1$, there is at least one other place $p' \in t^\bullet$ with $p' \neq p_1$. Thus

$$X[p_k] = \sum_{p \in t^\bullet} X[p] > X[p_1]\,.$$

Generalizing over $i \in \{1, \dots, k-1\}$, $X[p_i] \geq X[p_{i+1}]$,, implying $X[p_1] \geq X[p_k]$, which contradicts $X[p_k] > X[p_1]$. □

This result and the inequality of Theorem 4.2 together induce the following simple necessary and sufficient condition of liveness for ordinary well-formed strongly connected Join-Free systems, which are S-systems.

COROLLARY 4.4. *An ordinary, well-formed and strongly connected Join-Free system (S-system) $S = ((P, T, W), M_0)$ having at least one place and one transition is live if and only if $\sum_{p \in P} M_0(p) \geq 1$.*

### 4.4. Polynomial construction of a live marking for well-formed Join-Free systems

The next theorem, illustrated by Figure 11, is a specialization of the previous sufficient condition for all well-formed and weighted Join-Free systems. This theorem gives a construction of a class of live markings, a live marking for each choice of place $p_0$.

THEOREM 4.5. *Let $S = (N, M_0)$ be a weighted, strongly connected and conservative Join-Free system. $S$ is live if the following conditions hold:*

*– for a place $p_0$, $M_0(p_0) = max_{p_0}$*
*– for every place $p$ in $P - \{p_0\}$, $M_0(p) = max_p - gcd_p$.*

PROOF. By Theorem 3.8, there exists a balancing vector $\alpha \geq \mathbb{1}^{|P|}$ for $S$ such that $\alpha$ contains only positive natural numbers. Scaling $S$ by $\alpha$ yields the balanced system $S' = (N', M_0')$. Consequently,

$$\forall p \in P,\ max'_p - gcd'_p = \alpha_p \cdot (max_p - gcd_p)$$

with $max'_p - gcd'_p \in \mathbb{N}$. We deduce that

$$\sum_{p \in P} M_0'(p) \geq \sum_{p \in P - \{p_0\}} (max'_p - gcd'_p) + max'_{p_0} > \sum_{p \in P} (max'_p - gcd'_p)\,.$$

Moreover, for every place $p$, $max_p$ is a multiple of $gcd_p$. Thus $M_0'$ fulfills the useful tokens condition and by Theorem 4.2 the balanced Join-Free system $S'$ is live. According to Corollary 3.9, balancing a system preserves the property of liveness, thus $S$ is live. □

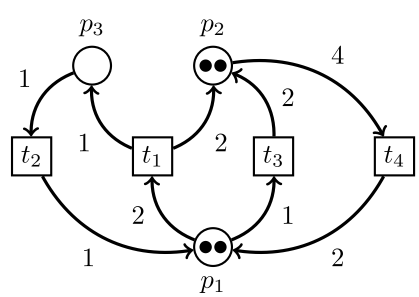



Fig. 11. The initial marking of this well-formed but not balanced Join-Free system fulfills the conditions of Theorem 4.5 and is thus well-behaved.

As a consequence of this result, we build a polynomial live initial marking for well-formed Choice-Free nets in the next section.

## 5. WELL-BEHAVEDNESS OF CHOICE-FREE SYSTEMS

In this section, we recall known structural properties of well-formedness and a property relating the liveness of a Choice-Free system to the liveness of particular subnets. Exploiting these properties and the results obtained in previous sections, we deduce a polynomial live initial marking for well-formed Choice-Free systems. We also deduce from these new results a known sufficient condition of liveness that was developed for T-systems [Marchetti and Munier-Kordon 2009].

### 5.1. Well-formedness, liveness and Fork-Attribution P-subnets

We recall properties of Choice-Free systems and present results leading to the polynomial live marking.

A *source* place is defined as a place with at least one output transition and without input transition. The liveness of a Choice-Free system can be stated by observing the liveness of some of its Fork-Attribution subsystems.

THEOREM 5.1 ([TERUEL ET AL. 1997]). *Let $S = (N, M_0)$ be a Choice-Free system without source places. $S$ is live if and only if the systems $S_{FA} = (N_{FA}, M_0[P_{FA}])$ are live for every strongly connected FA P-subnet $N_{FA}$ of $N$.*

Figure 4 represents all strongly connected FA P-subnets of the Choice-Free net shown in Figure 3. The idea is to provide a marking for the Choice-Free net that makes every strongly connected FA P-subnet live. We show that in the case of well-formed Choice-Free nets, all such FA P-subnets are well-formed and can benefit from our live marking for well-formed Join-Free nets.

Strongly connected and well-formed Choice-Free nets are consistent and conservative by Theorem 2.1. The following lemma shows that these properties propagate to P-subnets.

LEMMA 5.2. *All strongly connected FA P-subnets of a strongly connected well-formed Choice-Free net are consistent and conservative, hence well-formed.*

PROOF. We show first that the FA P-subnets are consistent and we deduce their conservativeness. Consider a well-formed and strongly connected, hence consistent, Choice-Free net $N$. By definition of consistency, there exists a vector $Y \geq \mathbb{1}^{|T|}$ such that $C \cdot Y = 0$ where $C$ is the incidence matrix of $N$. Thus, $Y$ annuls each row of $C$.

By definition, an FA P-subnet $N_{FA}$ of $N$ is composed of a subset of places $P_{FA}$ and the set of all their input and output transitions. Thus, the incidence matrix $C_{FA}$ of $N_{FA}$ is obtained by removing from $C$ the rows and columns that do not correspond to the places and transitions of $N_{FA}$. Since the vector $Y$ annuls all rows of $C$, its restriction to the transitions of $N_{FA}$ annuls all rows of $C_{FA}$ and $N_{FA}$ is consistent.

According to Theorem 2.1, if an FA P-subnet is consistent and strongly connected, then it is conservative and well-formed. □

Thus, the well-formedness of a Choice-Free net induces strong structural properties for its P-subnets. Morever, such subnets conform to balancing since they are conservative.

### 5.2. Polynomial construction of a live marking for well-formed Choice-Free systems

The subclass of Choice-Free nets allows transitions that have several input places. The firing of such transitions may occur only when all their input places are enabled. We base on the previous results on FA P-subnets to construct a polynomial live marking for well-formed Choice-Free systems.

*Definition* 5.3. A *multiple transition* is a transition with at least two input places.

When a Choice-Free net has no multiple transition, it is an FA net and the live marking of Theorem 4.5 applies. The interesting case deals with at least one multiple transition.

We first prove a technical lemma on the structure of the strongly connected FA P-subnets of well-formed Choice-Free nets.

LEMMA 5.4. *Let $N$ be a strongly connected and well-formed Choice-Free net with at least one multiple transition. Every strongly connected FA P-subnet of $N$ contains at least one input place of a multiple transition.*

PROOF. Suppose there exists a strongly connected FA P-subnet $N_{FA}$ containing only transitions having a unique input place in $N$. Then, either $N_{FA}$ is equal to $N$ which is an FA net, a contradiction, or $N_{FA}$ is a proper subnet and there exist a node $n$ in $N_{FA}$ and a node $n'$ in $N - N_{FA}$, such that $n'$ is an input of $n$, since $N$ is strongly connected. The node $n$ cannot be a place, otherwise $N_{FA}$ would not be a P-subnet. Hence $n$ is a transition with at least two input places: the one in $N_{FA}$, $N_{FA}$ being strongly connected, and $n'$. Thus $N_{FA}$ contains a transition that is multiple in $N$, a contradiction. □

THEOREM 5.5. *Let $S = (N, M_0)$ be a well-formed Choice-Free system which is not an FA system. $S$ is well-behaved if for all input places $p_i$ of all multiple transitions, $M_0(p_i) = max_{p_i}$ and for all other places $p$, $M_0(p) = max_p - gcd_p$ .*

PROOF. By Lemma 5.4, every strongly connected FA P-subnet contains at least an input place of a multiple transition, hence at least a place $p$ such that $M_0(p) = max_p$. Let $N_{FA} = (P_{FA}, T_{FA}, W_{FA})$ be any of the strongly connected FA P-subnets of $N$ and $S_{FA} = (N_{FA}, M_0^{FA})$, where $M_0^{FA}$ is the restriction of $M_0$ to $P_{FA}$. By Lemma 5.2, $N_{FA}$ is conservative and, by Theorem 4.5, $S_{FA}$ is live. Thus, the marking $M_0$ makes any strongly connected FA P-subnet live. $N_{FA}$ is well-formed thus without source place and, by Theorem 5.1, $(N, M_0)$ is live. □

Figure 12 pictures a well-behaved Choice-Free system. Indeed, this system is well-formed (see Fig. 5), the marking of each input place $p$ of a multiple transition equals $max_p$ and other places $p_i$ contain $max_{p_i} - gcd_{p_i}$ tokens.

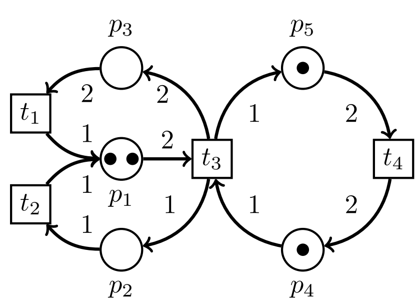



Fig. 12. The Choice-Free system is well-formed, thus the live initial marking of Theorem 5.5 makes the system well-behaved. In order to construct this live initialization, notice that $t_3$ is the only multiple transition, thus its input places $p_1$ and $p_4$ begin respectively with $max_{p_1} = 2$ and $max_{p_4} = 1$ tokens. It is then sufficient to mark every other place $p$ with $max_p - gcd_p$ tokens.

### 5.3. Special case of normalized T-systems

For T-systems, a sufficient condition of liveness was expressed in [Marchetti and Munier-Kordon 2009]. We prove here that this result can be viewed as a direct consequence of Theorem 4.2.

LEMMA 5.6. *All strongly connected FA P-subnets of a strongly connected T-net are circuits.*

PROOF. Let us consider a strongly connected FA P-subnet $N$ of a strongly connected T-net and suppose by contradiction that a transition $t$ of $N$ has at least two output places $p_1$ and $p_2$. Let $p$ be the unique input place of $t$. Places $p_1$ and $p_2$ belong to two different paths leading to $p$ since the net is strongly connected. These paths are not allowed to merge as two inputs of a place since the net is a T-net. They cannot be two inputs of a transition either, since the net is an FA net hence Join-Free, and $N$ is not strongly connected, a contradiction. □

We recall a known sufficient condition of liveness for T-systems that we prove now, differently from [Marchetti and Munier-Kordon 2009].

THEOREM 5.7 ([MARCHETTI AND MUNIER-KORDON 2009]).
*Let $S = ((P, T, W), M_0)$ be a strongly connected and normalized T-system that fulfills the useful tokens condition. $S$ is live if every circuit $S_C$ of $S$ satisfies*

$$\sum_{p \in P_C} M_0(p) > \sum_{p \in P_C} (max_p - gcd_p) \,,$$

*where $P_C$ is the set of places of the circuit $S_C$.*

PROOF. By definition of the normalization, if $(P, T, W)$ is a normalized T-net then its circuits are balanced. Moreover, balanced circuits are strongly connected and balanced FA nets, thus Theorem 4.2 applies. T-nets are Choice-Free nets and $S$ is strongly connected hence without source place. By Lemma 5.6 and Theorem 5.1, the claim is proved. □

## 6. SUFFICIENT CONDITIONS ARE NOT NECESSARY

All previous sufficient conditions of liveness for Join-Free and Choice-Free systems are not necessary. This is shown through a counter example that comes from the T-system [Marchetti and Munier-Kordon 2009] pictured in Figure 13. It consists of a live marked circuit that does not fulfill the sufficient conditions. Indeed,

$$\sum_{p} (max_p - gcd_p) = (14 - 2) + (21 - 7) + (6 - 3) = 29 \,.$$

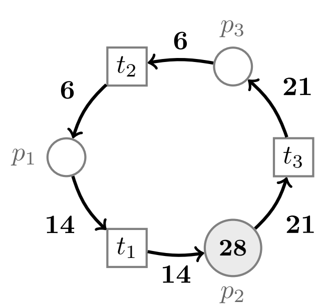



Fig. 13. This circuit is a live Join-Free and Choice-Free system but does not fulfill their sufficient conditions.

The reachability graph of Figure 14 shows that every transition can be fired from any reachable marking after a finite firing sequence, thus the circuit of Figure 13 is live.

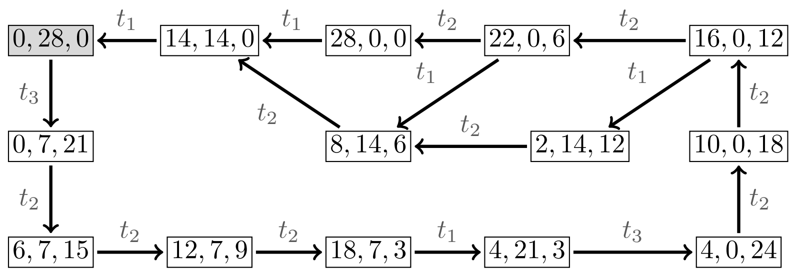



Fig. 14. The reachability graph of the circuit shows all its feasible sequences. Starting from any reachable marking, there exists a finite feasible sequence containing every transition, implying liveness.

## 7. HOME MARKING FOR FORK-ATTRIBUTION SYSTEMS

Fork-Attribution systems form a subclass of Choice-Free systems from which they inherit their structural properties. Strongly connected and well-formed Choice-Free nets are shown in [Teruel et al. 1997] to have exactly one minimal T-semiflow, from which it follows that such a T-semiflow also exists for strongly connected and well-formed Fork-Attribution nets. Moreover, an initial marking is a home marking if and only if there exists a firing sequence whose Parikh vector is equal to the minimal T-semiflow.

We study properties of the sequences of firings that are smaller than the minimal T-semiflow in FA systems. We use these properties to show that the live marking of Theorem 4.5 allows to fire a sequence whose Parikh vector is equal to this T-semiflow. We deduce that this live marking is also a home marking for FA systems.

THEOREM 7.1 ([TERUEL ET AL. 1997]). *If $N = (P, T, W)$ is a well-formed and strongly connected Choice-Free net, then $N$ has a unique minimal T-semiflow $Y$. Moreover, the support of $Y$ is the whole set $T$.*

The following theorem provides a characterization of the home markings in well-behaved Choice-Free systems.

THEOREM 7.2 ([TERUEL ET AL. 1997]). *Consider a well-formed, live and strongly connected Choice-Free system $S = (N, M_0)$ with unique minimal T-semiflow $Y$. $M_0$ is a home marking if and only if a sequence $\sigma_Y$ such that $\vec{\sigma}_Y = Y$ is feasible at $M_0$.*

As we focus on strongly connected and well-formed Fork-Attribution systems, we consider only systems having such a unique minimal T-semiflow whose support is the set of all transitions. The following definitions apply to all nets having this structural property.

*Definition* 7.3. Consider a system with a unique minimal T-semiflow $Y$. A feasible sequence $\sigma$ is *transient* if its Parikh vector is strictly smaller than $Y$, that is $\vec{\sigma}(t) \leq Y(t)$ for every transition and $\vec{\sigma}(t) < Y(t)$ for at least one transition.

*Definition* 7.4. Consider a transient sequence $\sigma$ for a system with a unique minimal T-semiflow $Y$. A *transition* $t$ is *ready* for $\sigma$ if $\vec{\sigma}(t) < Y(t)$, otherwise the transition is *frozen*, in which case $\vec{\sigma}(t) = Y(t)$. A *place* is *ready* for $\sigma$ if it is an input of a ready transition for $\sigma$, otherwise it is *frozen*.

As the following lemma shows, a frozen place cannot contain more than its initial number of tokens.

LEMMA 7.5. *Consider a strongly connected and well-formed FA system denoted by $S = ((P, T, W), M_0)$ and a transient sequence $\sigma$ such that $M_0 \xrightarrow{\sigma} M$ and there exists a* *frozen place $p$ for $\sigma$. If all input transitions of $p$ are frozen, then $M(p) = M_0(p)$, otherwise $M(p) < M_0(p)$.*

PROOF. The sequence $\sigma$ is transient thus $\vec{\sigma}(t) \leq Y(t)$ for every transition $t$. Besides, $p$ is frozen thus its unique output transition $t$ has fired exactly $Y(t)$ times: $\vec{\sigma}(t) = Y(t)$, and every input transition $t_i$ of $p$ has fired at most $Y(t_i)$ times.

We deduce that

$$M(p) = M_0(p) + \sum_{t_i \in {}^\bullet p} W(t_i, p) \cdot \vec{\sigma}(t_i) - W(p,t) \cdot \vec{\sigma}(t)$$
$$\leq M_0(p) + \sum_{t_i \in {}^\bullet p} W(t_i, p) \cdot Y(t_i) - W(p,t) \cdot Y(t) = M_0(p)\,.$$

The equality $M(p) = M_0(p)$ is obtained when input transitions $t_i$ are frozen and thus have been fired $Y(t_i)$ times. The inequality is strict when at least one of them has been fired a strictly smaller number of times. □

LEMMA 7.6. *Consider a strongly connected and well-formed FA system $S_{FA}$ and a transient sequence $\sigma$ for $S_{FA}$. If there exist at least one ready place and one frozen place for $\sigma$ then there exists a frozen place with at least one ready input for $\sigma$.*

PROOF. As $S_{FA}$ is strongly connected, there exists a circuit containing the frozen place and the ready place. Moreover, the unique output transition of any ready place is ready. Consequently, there exists a frozen place having at least one ready input transition for $\sigma$. □

We show next that for a well-formed FA net initially marked by $\widehat{M_0}$, the live marking of Theorem 4.5, there is at least one enabled ready place after any transient sequence.

LEMMA 7.7. *Consider a strongly connected and well-formed FA system $S_{FA} = ((P,T,W), \widehat{M_0})$ and a transient sequence $\sigma$ such that $\widehat{M_0} \xrightarrow{\sigma} \widehat{M}$. Then there exists at least one enabled ready place at $\widehat{M}$ for $\sigma$.*

PROOF. On one hand, since $\sigma$ is transient, there exists at least one ready place at $\widehat{M}$ for $\sigma$. On the other hand, since the system $S_{FA}$ is live by Theorem 4.5 and places have exactly one output transition, there exists at least one enabled place at $\widehat{M}$.

The claim of the theorem is clearly true if all places are ready for $\sigma$. The remainder of the proof deals with the case where some places are not ready for $\sigma$.

The system $S_{FA}$ is conservative hence, according to Theorem 3.8, it can be balanced. Let $S'_{FA} = ((P,T,W'), \widehat{M}'_0)$ be a balancing of $S_{FA}$. The sequence $\sigma$ is feasible in $S'_{FA}$ by Theorem 3.2, hence there exists $\widehat{M}'$ such that $\widehat{M}'_0 \xrightarrow{\sigma} \widehat{M}'$. By Theorem 3.4, $S_{FA}$ and $S'_{FA}$ have the same unique minimal T-semiflow $Y$, hence $\sigma$ is transient in both systems and, consequently, $\widehat{M}$ and $\widehat{M}'$ have the same sets of ready and frozen places.

Thus, there is at least one ready place in $S'_{FA}$ at $\widehat{M}'$ for $\sigma$.

Considering the transient sequence $\sigma$, we denote by $P_f$ the set of the frozen places having only frozen inputs (potentially empty), $P_{fr}$ the set of the frozen places having at least one ready input (not empty by assumption and by Lemma 7.6) and $P_r$ the set of the ready places (not empty by assumption). As depicted in Figure 15, $P_f$, $P_{fr}$ and $P_r$ form a tri-partition of the places of $S_{FA}$ and $S'_{FA}$:

$$\sum_{p \in P} \widehat{M}'(p) = \sum_{p \in P_f} \widehat{M}'(p) + \sum_{p \in P_{fr}} \widehat{M}'(p) + \sum_{p \in P_r} \widehat{M}'(p)\,,$$

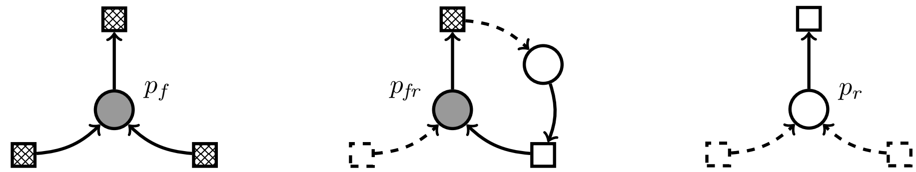



Fig. 15. The frozen nodes are shaded, while the ready nodes are white. The dashed nodes can be either frozen or ready. The places of the transient system are tri-partitioned into frozen places having only frozen inputs (the set $P_f$), frozen places having at least one ready input (the set $P_{fr}$) and ready places (the set $P_r$). The number of tokens in a frozen place $p$ is bounded above by $\widehat{M}_0'(p)$. As the number of tokens in the system remains constant, this tri-partition provides a lower bound of the total number of tokens in ready places.

hence

$$\sum_{p \in P_r} \widehat{M}'(p) = \sum_{p \in P} \widehat{M}'(p) \quad - \sum_{p \in P_f \cup P_{fr}} \widehat{M}'(p)\,. \tag{3}$$

This is also true for the initial marking:

$$\sum_{p \in P_r} \widehat{M}_0'(p) = \sum_{p \in P} \widehat{M}_0'(p) \quad - \sum_{p \in P_f \cup P_{fr}} \widehat{M}_0'(p)\,. \tag{4}$$

Since the number of tokens remains constant in $S'_{FA}$, $\sum_{p \in P} \widehat{M}_0'(p) = \sum_{p \in P} \widehat{M}'(p)$, and the first terms on the right-hand side of the above two equations are equal.

By Lemma 7.5,

$$\textstyle\sum_{p \in P_f} \widehat{M}'(p) = \sum_{p \in P_f} \widehat{M}_0'(p)$$

and, since $P_{fr}$ is not empty,

$$\textstyle\sum_{p \in P_{fr}} \widehat{M}'(p) < \sum_{p \in P_{fr}} \widehat{M}_0'(p)\,,$$

hence the last terms on the right-hand side of the equations satisfy

$$\textstyle\sum_{p \in P_f \cup P_{fr}} \widehat{M}'(p) < \sum_{p \in P_f \cup P_{fr}} \widehat{M}_0'(p)\,.$$

Thus the right-hand side of eqn. (4) is strictly smaller than the right-hand side of eqn. (3) and

$$\sum_{p \in P_r} \widehat{M}'(p) > \sum_{p \in P_r} \widehat{M}_0'(p) \geq \sum_{p \in P_r} (max_p' - gcd_p')\,,$$

where the last inequality follows from the definition of $\widehat{M}_0'$.

Recall that the tokens of $\widehat{M}'$ are all useful tokens, since $\widehat{M}_0'$ contains only useful tokens. Thus, at least one place of $P_r$ is then enabled by $\widehat{M}'$, having $max_p'$ or more tokens in $\widehat{M}'$. The language of the system is preserved (Theorem 3.2) and every feasible sequence is transient relatively to the same minimal T-semiflow (Theorem 3.4), thus ready places enabled by $\widehat{M}'$ in $S'_{FA}$ for $\sigma$ are also ready and enabled by $\widehat{M}$ in $S_{FA}$ for $\sigma$. □

The following result states that $\widehat{M_0}$ is a live home marking for every strongly connected and well-formed FA system.

THEOREM 7.8. *If $N_{FA}$ is a weighted, strongly connected and conservative FA net, then the system $S_{FA} = (N_{FA}, \widehat{M_0})$ is live and reversible.*

PROOF. The system is live by Theorem 4.5. According to Lemma 7.7, for any transient sequence $\sigma$ with $\widehat{M_0} \xrightarrow{\sigma} \widehat{M}$, at least one ready place is enabled by $\widehat{M}$. Moreover, every transition has only one input place. Thus, a ready transition is enabled by $\widehat{M}$ and a sequence whose Parikh vector equals the minimal T-semiflow of $S_{FA}$ can be fired exactly. By Theorem 7.2, the claim is proved. □

An extension of this proof to all well-formed Choice-Free nets would show that after any transient sequence, a ready transition is enabled, even if multiple transitions only are ready. However, it is not obvious that the coverability of the Choice-Free net by live and reversible strongly connected FA P-subnets induces the existence, after any transient sequence, of a ready transition whose every input place is enabled. Thus, the proof cannot be easily extended to well-formed Choice-Free nets.

Well-formed Join-Free nets do not benefit from a characterization of home markings such as the one given by Theorem 7.2 for Choice-Free nets. Thus, the proof cannot be applied to Join-Free nets either.

## 8. CONCLUSION

We proved that every conservative system can be made token-conservative, preserving the language of the system while the number of tokens remains constant at every reachable marking. This allows to simplify the study of behavioral properties of conservative systems, including liveness and reversibility. As the conservativeness property is known to be necessary for well-formedness, this transformation, called balancing, applies to all well-formed weighted Petri nets.

We illustrated the balancing on particular subclasses of weighted Petri nets, namely well-formed Join-Free and Choice-Free nets. For these two classes, we obtained the first polynomial sufficient conditions of liveness. Moreover, we presented the first polynomial live and home marking for the subclass of well-formed Fork-Attribution nets. All these conditions and markings have a polynomial time complexity, and require only a linear number of initial tokens, whereas prior methods used exponential time algorithms or an exponential number of initial tokens.

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