Analysis and Synthesis of Weighted Marked Graph Petri Nets

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Abstract. Numerous real-world systems can be modeled with Petri nets, which allow a combination of concurrency with synchronizations and conflicts. To alleviate the difficulty of checking their behaviour, a common approach consists in studying specific subclasses. In the converse problem of Petri net synthesis, a Petri net of some subclass has to be constructed efficiently from a given specification, typically from a labelled transition system describing the behaviour of the desired net.
In this paper, we focus on a notorious subclass of persistent Petri nets, the weighted marked graphs (WMGs), also called generalised (or weighted) event (or marked) graphs or weighted T-nets. In such nets, edges have multiplicities (weights) and each place has at most one ingoing and one outgoing transition. Although extensively studied in previous works and benefiting from strong results, both their analysis and synthesis can be further investigated. To this end, we provide new conditions delineating more precisely their behaviour and give a dedicated synthesis procedure.

Keywords: Weighted Petri net, analysis, synthesis, marked graph, event graph.

1 Introduction

Petri nets have proved useful to model numerous artificial and natural systems. Their weighted version allows weights on arcs, making possible the bulk consumption or production of tokens, hence a more compact representation of the systems.

Many fundamental properties of Petri nets are decidable, although often hard to check. Given a bounded Petri net, a naive analysis can be performed by constructing its finite reachability graph, whose size may be considerably larger than the net size. To avoid such a costly computation, subclasses are often considered, allowing to derive efficiently their behaviour from their structure only. This approach has led to various polynomial-time checking methods dedicated to several subclasses, the latter being defined by structural restrictions in many cases [13, 28, 24, 17, 18].

In the domain of Petri net synthesis, a specification has to be implemented by a Petri net, meaning that the behaviour of the Petri net obtained must correspond exactly to the specification. Classical representations of such a specification encompass labelled transitions systems (lts for short), which are rooted directed graphs with labels on the arcs, and a synthesis procedure is meant to build a Petri net of a specific subclass whose reachability graph is isomorphic to a given lts.

Weighted marked graphs: applications and previous studies. In this paper, we focus on marked graphs with weights (also called generalised event graphs and weighted T-nets), a subclass of weighted Petri nets in which each place has at most one input and one output. They can model Synchronous DataFlow graphs [21], which have been fruitfully used to design and analyse many real systems such as embedded applications, notably Digital Signal Processing (DSP) applications [20, 25, 23].

Various characterisations and polynomial-time sufficient conditions of structural and behavioural properties, notably of liveness, boundedness and reversibility, have been developed for this class [26, 22]. These nets are a special case of persistent systems, in which no transition firing can disable another transition.

Petri net synthesis: previous studies. Given a labelled transition system, previous works have proposed algorithms synthesizing a Petri net with an isomorphic reachability graph, sometimes aiming at a Petri net subclass [6, 9]. In the latter case, the objective is to delineate properties of the lts that are specific to the target subclass, so as to determine sufficient and necessary conditions for its synthesisability within the subclass. Ideally, such specific conditions should be easier to check than generic ones, for instance during a pre-synthesis phase.

Marked graphs, i.e. unit-weighted marked graphs, belong to the larger class of choice-free nets, in which each place has at most one output. Both classes benefit from dedicated synthesis algorithms that operate in polynomial time [2, 4, 8, 6, 9]. However, such methods do not yet exist for the intermediate class of marked graphs with arbitrary weights.

Contributions. In this paper, we further investigate the class of weighted marked graphs (WMGs). We delineate new properties of these nets and propose a synthesis procedure aiming at this subclass.

First, we provide new structural and behavioural properties of WMGs: we give a comparison property on the sequences starting at the same state and reaching another common state, we show that WMGs are necessarily *backward persistent*, meaning that for all reachable states s_1, s_2, s_3 such that $s_2[a\rangle s_1$ (i.e. s_1 is reached from s_2 through the action with label a) and $s_3[b\rangle s_1$, there exists a reachable state s_4 with $s_4[b\rangle s_2$ and $s_4[a\rangle s_3$. We also develop conditions allowing the existence of a feasible sequence corresponding to a given Parikh vector.

Then, we delineate necessary conditions for the WMG-solvability of an lts, such as backward persistence and the existence of particular cycles. We show, with the help of a counter-example from another subclass, that these conditions are not sufficient for a WMG solution to exist.

Finally, we devise a WMG-synthesis procedure, specialising previous methods that were designed for the larger class of choice-free nets.

Organisation of the paper. In Section 2, we introduce general definitions, notations and properties. In Section 3, we recall some properties of persistent Petri nets and provide new structural and behavioural results on WMGs, including the proof of backward persistence. In Section 4, we describe a synthesis procedure for WMGs. Section 5 presents our conclusion with perspectives.

2 Classical Definitions, Notations and Properties

In the following, we define formally Petri nets, labelled transitions systems and related notions. We also recall classical properties of Petri nets in Proposition 1.

Petri nets, incidence matrices, pre- and post-sets. A *(Petri) net* is a tuple $N = (P, T, W)$ such that P is a finite set of *places*, T is a finite set of *transitions*, with $P \cap T = \emptyset$, and W is a weight function $W\colon ((P \times T) \cup (T \times P)) \to \mathbb{N}$ setting the weights on the arcs. A *marking* of the net N is a mapping from P to $\mathbb{N}$, i.e. a member of $\mathbb{N}^P$, defining the number of tokens in each place of N.
A *(Petri net) system* is a tuple $\zeta = (N, M_0)$ where N is a net and M_0 is a marking, often called *initial marking*. The *incidence matrix* C of N (and ζ) is the integer place-transition matrix with components $C(p,t) = W(t,p) - W(p,t)$, for each place p and each transition t.
The *post-set* $n^\bullet$ and *pre-set* ${}^\bullet n$ of a node $n \in P \cup T$ are defined as $n^\bullet = \{n' \in P \cup T \mid W(n,n')>0\}$ and ${}^\bullet n = \{n' \in P \cup T \mid W(n',n)>0\}$.

Firings and reachability in Petri nets. Consider a system $\zeta = (N, M_0)$ with $N = (P, T, W)$. A transition t is *enabled* at M_0 (i.e. in ζ) if $\forall p \in {}^\bullet t$, $M_0(p) \geq W(p,t)$, in which case t can *occur* at or be *fired* from M_0. The firing of t from M_0 leads to the marking $M = M_0 + C[P,t]$ where $C[P,t]$ is the column of C associated to t: we note this as $M_0[t\rangle M$.
A finite *(firing) sequence* σ of length $n \geq 0$ on the set T, denoted by $\sigma = t_1 \dots t_n$ with $t_1 \dots t_n \in T$, is a mapping $\{1, \dots, n\} \to T$. Infinite sequences are defined similarly as mappings $\mathbb{N} \setminus \{0\} \to T$. A sequence σ of length n is *enabled* in ζ if the successive states obtained, $M_0[t_1\rangle M_1 \dots [t_n\rangle M_n$, satisfy $M_{k-1}[t_k\rangle M_k$, $\forall k \in \{1, \dots, n\}$, in which case M_n is said to be *reachable* from M_0: we note this as $M_0[\sigma\rangle M_n$. If $n = 0$, σ is the *empty sequence* ϵ, implying $M_0[\epsilon\rangle M_0$. The set of markings reachable from M_0 is noted $[M_0\rangle$.
The *reachability graph* of ζ, noted $RG(\zeta)$, is the rooted directed graph (V, A, ι) where V represents the set of vertices $[M_0\rangle$, A is the set of arcs labelled with transitions of T such that the arc $M \xrightarrow{t} M'$ belongs to A if and only if $M[t\rangle M'$ and $M \in [M_0\rangle$, and $\iota = M_0$ is the root.

In Figure 1, a weighted system is pictured on the left. Its reachability graph is pictured on the right, where v^T denotes the transpose of vector v.

Petri net subclasses. N is *plain* if no arc weight exceeds 1; *choice-free* (CF for short) [11, 27] (also called *place-output-nonbranching* in [5]) if $\forall p \in P$, $|p^\bullet| \leq 1$; *fork-attribution* (FA) [27] if it is CF and, in addition, $\forall t \in T$, $|^\bullet t| \leq 1$; a *weighted marked graph* (WMG, also called weighted T-system in [26]) if it is CF and, in addition, $\forall p \in P$, $|^\bullet p| \leq 1$. A WMG is pictured on the left of Figure 1. Well-studied subclasses encompass *marked graphs* [10], which are plain and fulfill $|p^\bullet| = 1$ and $|^\bullet p| = 1$ for each place p, and *T-systems* [13], which are plain and fulfill $|p^\bullet| \leq 1$ and $|^\bullet p| \leq 1$ for each place p.

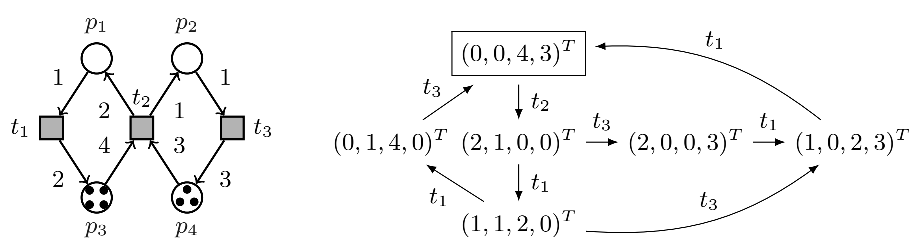


Fig. 1. A WMG system ζ and its reachability graph $RG(\zeta)$ are pictured respectively on the left and on the right. The initial marking is boxed in $RG(\zeta)$.

Lts and their relationship with Petri nets. A *labelled transition system with initial state*, abbreviated *lts*, is a quadruple $TS = (S, \rightarrow, T, \iota)$ where S is the set of *states*, T is the set of *labels*, $\rightarrow \subseteq (S \times T \times S)$ is the *transition relation*, and $\iota \in S$ is the *initial state*.
A label t is *enabled* at $s \in S$ if $\exists s' \in S \colon (s, t, s') \in \rightarrow$, written $s[t\rangle$ or $s[t\rangle s'$, in which case s' is *reachable* from s through the execution of t. We denote by $s^\bullet$ the set $\{s' | \exists t \in T, s[t\rangle s'\}$.
A label t is *backward enabled* at s if $\exists s' \in S \colon (s', t, s) \in \rightarrow$, written $[t\rangle s$ or $s'[t\rangle s$. A *(firing) sequence* σ of length $n \geq 0$ on the set of labels T, denoted by $\sigma = t_1 \dots t_n$ with $t_1 \dots t_n \in T$, is *enabled* at some state s_0 if the successive states obtained, $s_0[t_1\rangle s_1 \dots [t_n\rangle s_n$, satisfy $s_{k-1}[t_{i_k}\rangle s_k$, $\forall k \in \{1, \dots, n\}$: we note $s_0[\sigma\rangle s_n$. Similarly, other notions and notations, related to sequences and reachability in Petri nets, extend readily to labelled transition systems by replacing markings with states.
The reachability graph $RG(\zeta)$ of a system $\zeta = (N, M_0)$ can be represented by the labelled transition system $TS = (S, \rightarrow, T, \iota)$ if an isomorphism $\gamma \colon S \rightarrow [M_0\rangle$ exists such that $\gamma(\iota) = M_0$ and $(s, t, s') \in \rightarrow \Leftrightarrow \gamma(s)[t\rangle\gamma(s')$ for all $s, s' \in S$. If an lts TS is isomorphic to the reachability graph of a Petri net system ζ, we say that ζ *solves* TS, and that it WMG-solves TS if N is a WMG.
These notions are illustrated on the lts on the right of Figure 2, which is isomorphic to the reachability graph of the WMG in Figure 1; it is thus WMG-solvable.

Two lts $TS_1 = (S_1, \rightarrow_1, T, s_{01})$ and $TS_2 = (S_2, \rightarrow_2, T, s_{02})$ are isomorphic if there is a bijection $\beta\colon S_1 \rightarrow S_2$ with $\beta(s_{01}) = s_{02}$ and $(s, t, s') \in\rightarrow_1 \Leftrightarrow (\beta(s), t, \beta(s')) \in\rightarrow_2$, for all $s, s' \in S_1$.

Vectors, semiflows and cycles. The *support* of a vector is the set of the indices of its non-null components. Consider any net $N = (P, T, W)$ with its incidence matrix C. A *T-vector* is an element of $\mathbb{N}^T$; it is called *prime* if the greatest common divisor of its components is one (i.e. its components do not have a common non-unit factor). A *T-semiflow* ν of the net is a non-null T-vector whose components are only non-negative integers (i.e. $\nu \gneqq \mathbb{0}$) and such that $C \cdot \nu = \mathbb{0}$. A T-semiflow is called *minimal* when it is prime and its support is not a proper superset of the support of any other T-semiflow [27].
The *Parikh vector* $\mathbf{P}(\sigma)$ of a finite sequence σ of transitions is a T-vector counting the number of occurrences of each transition in σ, and the *support* of σ is the support of its Parikh vector, i.e. $supp(\sigma) = supp(\mathbf{P}(\sigma)) = \{t \in T \mid \mathbf{P}(\sigma)(t) > 0\}$. A (non-empty) cycle around a marking M is a non-empty sequence σ such that $M[\sigma\rangle M$; the Parikh vector of a non-empty cycle is a T-semiflow and a non-empty cycle is called prime if its Parikh vector is prime.

Further notions. Consider a lts $TS = (S, \rightarrow, T, \iota)$. For all states $s, s' \in S$, a sequence $s[\sigma\rangle s'$ is called a *cycle*, or more precisely a *cycle at (or around) state* s, if $s = s'$. A non-empty cycle $s[\sigma\rangle s$ is called *small* if there is no non-empty cycle $s'[\sigma'\rangle s'$ in TS with $\mathbf{P}(\sigma') \lneqq \mathbf{P}(\sigma)$. A *two-way uniform chain* of TS is a couple $(\{s_i \in S | i \in \mathbb{Z}, \forall i, j \in \mathbb{Z} : i \neq j \Rightarrow s_i \neq s_j\}, \sigma \in T^+)$ such that $\forall i \in \mathbb{Z}, s_i[\sigma\rangle s_{i+1}$, where T^+ is the set of non-empty sequences on T.

In Figure 2, a two-way uniform chain is depicted on the left; on the right, the lts is finite, hence has no two-way uniform chain. The lts TS is:

- *totally reachable* if $S = [\iota\rangle$;
- *reversible* if $\iota \in [s\rangle$ for each state $s \in [\iota\rangle$, meaning the strong connectedness of this lts when it is totally reachable;
- *weakly periodic* if for each couple $(\{s_i \in S | i \in \mathbb{N}\}, \sigma \in T^+)$ such that $\forall i \in \mathbb{N}$ $s_i[\sigma\rangle s_{i+1}$ (where σ is a non-empty sequence of labels), either $s_i = s_j$ $\forall i, j \in \mathbb{N}$, or $i \neq j \Rightarrow s_i \neq s_j$ $\forall i, j \in \mathbb{N}$;
- *strongly cycle consistent* if for every sequence $s[\alpha\rangle s'$, the existence of cycles $s_1[\beta_1\rangle s_1$, $s_2[\beta_2\rangle s_2$, $\ldots$, $s_n[\beta_n\rangle s_n$ and of numbers $k_1, k_2, \ldots, k_n \in \mathbb{Q}$ such that $\mathbf{P}(\alpha) = \sum_{i=1}^{n} k_i \cdot \mathbf{P}(\beta_i)$ implies that $s = s'$;
- *deterministic* if, for all states $s, s', s'' \in S$ and labels $t, t' \in T$ such that $s[t\rangle s' \wedge s[t\rangle s''$, necessarily $s' = s''$; it is *fully deterministic* if for all sequences σ and σ' such that $\mathbf{P}(\sigma) = \mathbf{P}(\sigma')$, we have, for all states $s, s', s'' \in S$: $s[\sigma\rangle s' \wedge s[\sigma'\rangle s'' \Rightarrow s' = s''$;
- *backward deterministic* if, for all states $s, s', s'' \in S$ and labels $t, t' \in T$ such that $s'[t\rangle s \wedge s''[t\rangle s$, necessarily $s' = s''$; it is *fully backward deterministic* if, for all sequences σ and σ' such that $\mathbf{P}(\sigma) = \mathbf{P}(\sigma')$, we have, for all states $s, s', s'' \in S$: $s'[\sigma\rangle s \wedge s''[\sigma'\rangle s \Rightarrow s' = s''$;

- *persistent* if for all states $s, s', s'' \in S$ and labels $t', t'' \in T$ such that $s[t'\rangle s'$ and $s[t''\rangle s''$ with $t' \neq t''$, there exists a state $s''' \in S$ such that $s'[t''\rangle s'''$ and $s''[t'\rangle s'''$; it is *backward persistent* if for all states $s, s', s'' \in S$ and labels $t', t'' \in T$ such that $s'[t'\rangle s$ and $s''[t''\rangle s$ with $t' \neq t''$, there exists a state $s''' \in S$ such that $s'''[t''\rangle s'$ and $s'''[t'\rangle s''$.

Figure 2 illustrates some of these notions. All notions defined for labelled transition systems apply to Petri nets through their reachability graphs. For example, a Petri net is reversible if its reachability graph is isomorphic to a reversible lts, meaning that the initial marking is reachable from every reachable marking.

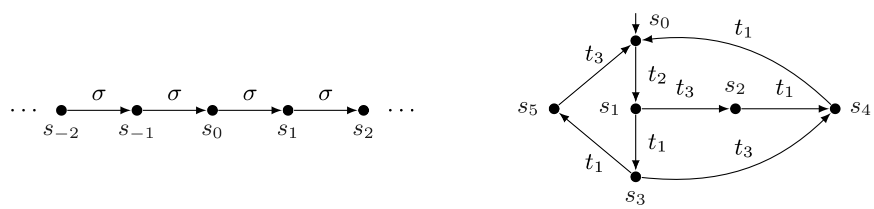


Fig. 2. On the left, a two-way uniform chain based on σ. On the right, a labelled transition system with states $\{s_0, s_1, s_2, s_3, s_4, s_5\}$, labels $\{t_1, t_2, t_3\}$ and initial state $\iota = s_0$. It is isomorphic to the reachability graph of Figure 1. The label t_2 is enabled at s_0 and t_3 is backward enabled at s_0. The state s_1 is reachable from s_0 through the execution of t_2. Denote by σ the sequence $t_2 t_3 t_1 t_1$. Then, the Parikh vector of σ is $\mathbf{P}(\sigma) = (2, 1, 1)$ and its support is $supp(\sigma) = \{t_1, t_2, t_3\}$. Since $s_0[\sigma\rangle s_0$, σ is a cycle around state s_0. This lts is totally reachable, weakly periodic, fully deterministic and fully backward deterministic, strongly cycle consistent, persistent, backward persistent and reversible.

The following proposition recalls properties satisfied by every Petri net system and presented in [5].

Proposition 1 (Classical properties of Petri nets [5]). *If $\zeta = (N, M_0)$, where $N = (P, T, W)$, is a Petri net system, then $RG(\zeta)$ is totally reachable, weakly periodic, fully deterministic, fully backward deterministic, and strongly cycle consistent. Moreover it has no two-way uniform chain over the set $S = \mathbb{N}^P$ of all the possible markings for N, meaning that no couple $(\{M_i \in \mathbb{N}^P | i \in \mathbb{Z} \setminus \{0\}, \forall i, j \in \mathbb{Z} : i \neq j \Rightarrow s_i \neq s_j\}, \sigma \in T^+)$ exists such that $\forall i \in \mathbb{Z}$, $M_i[\sigma\rangle M_{i+1}$.*

3 Properties of WMGs and Larger Persistent Classes

In this section, we investigate the structure and behaviour of WMGs. For that purpose, we first recall notions and results relevant to persistent systems in Subsection 3.1. Then, in Subsection 3.2, for the class of WMGs, we show a property of the sequences sharing the same starting state and the same ending state, and we prove backward persistence. Finally, in Subsection 3.3, we propose conditions for the existence of feasible sequences corresponding to a given T-vector in WMGs.

3.1 Previous Results and Notions Related to Persistence

In addition to the general properties of Petri nets mentioned in Proposition 1, we recall results and notions useful to the study of persistent systems.

The next result is dedicated to WMGs and extracted from [26, 27].

Proposition 2 (Minimal T-semiflow and cycles in WMGs [26, 27]). *Consider a connected WMG net N. If N has a T-semiflow ν then there exists a unique minimal (hence prime) one π, which satisfies: $supp(\pi) = T$ and $\nu = k \cdot \pi$ for some integer $k > 0$. Moreover, for any marking M_0, writing $\zeta = (N, M_0)$, if $RG(\zeta)$ contains some non-empty cycle, then the Parikh vector of each small cycle of $RG(\zeta)$ equals π.*

The next notion of residues is useful to the study of persistent systems.

Definition 1 (Residues). *Let T be a set of labels and $\tau, \sigma \in T^*$ two sequences over this set. The* (left) residue of τ with respect to σ, *denoted by $\tau \stackrel{\bullet}{-} \sigma$, arises from cancelling successively in τ the leftmost occurrences of all symbols from σ, read from left to right. Inductively: $\tau \stackrel{\bullet}{-} \varepsilon = \tau$; $\tau \stackrel{\bullet}{-} t = \tau$ if $t \notin supp(\tau)$; $\tau \stackrel{\bullet}{-} t$ is the sequence obtained by erasing the leftmost t in τ if $t \in supp(\tau)$; and $\tau \stackrel{\bullet}{-} (t\sigma) = (\tau \stackrel{\bullet}{-} t) \stackrel{\bullet}{-} \sigma$. For example, $acbcacbc \stackrel{\bullet}{-} abbcb = cacc$ and $abbcb \stackrel{\bullet}{-} acbcacbc = b$.*

We deduce the next property of residues.

Lemma 1 (Disjoint support of residues). *For any two sequences τ and σ, the residues $\delta_1 = \tau \stackrel{\bullet}{-} \sigma$ and $\delta_2 = \sigma \stackrel{\bullet}{-} \tau$ have disjoint supports: $supp(\delta_1) \cap supp(\delta_2) = \emptyset$. Consequently, $\delta_1 \stackrel{\bullet}{-} \delta_2 = \delta_1$ and $\delta_2 \stackrel{\bullet}{-} \delta_1 = \delta_2$.*

Proof. For any label t, $\mathbf{P}(\tau)(t) = \mathbf{P}(\sigma)(t) \Rightarrow \mathbf{P}(\delta_1)(t) = \mathbf{P}(\delta_2)(t) = 0$, $\mathbf{P}(\tau)(t) > \mathbf{P}(\sigma)(t) \Rightarrow \mathbf{P}(\delta_2)(t) = 0$ and $\mathbf{P}(\tau)(t) < \mathbf{P}(\sigma)(t) \Rightarrow \mathbf{P}(\delta_1)(t) = 0$. In all cases, $t \notin supp(\delta_1) \cap supp(\delta_2)$. □

Kellers's theorem is based on residues and applies to persistent lts.

Theorem 1 (Keller [19]). *Let $(S, \rightarrow, T, \iota)$ be a deterministic, persistent lts. Let τ and σ be two label sequences enabled at some state s. Then $\tau(\sigma \stackrel{\bullet}{-} \tau)$ and $\sigma(\tau \stackrel{\bullet}{-} \sigma)$ are both enabled at s and lead to the same state.*

Applying Theorem 1, we obtain the next result directly.

Proposition 3 (Persistence and determinism). *Let TS be a persistent lts. If TS is also deterministic, then it is fully deterministic.*

3.2 Equivalent Sequences and Backward Persistence

In the following, we provide new properties on the reachability graph of WMGs. Since non-connected nets can be studied by analysing each connected component separately, we restrict our attention to connected nets.

For the class of WMGs, we first provide in Lemma 2 a property of the sequences starting from a same state s and leading to the same state s'. Then, we prove the backward persistence of WMGs in Theorem 2. To achieve it, we need to define the reverse of a net and of a firing sequence.

Definition 2 (Reverse nets and sequences). *The* reverse *of a net N, denoted by $-N$, is obtained from N by reversing all the arcs while keeping the weights. We denote by σ^{-1} the sequence σ followed in reverse order. For example, if $\sigma = t_1 t_2 t_2 t_3$, then $\sigma^{-1} = t_3 t_2 t_2 t_1$.*

The set of WMGs is closed under reverse, contrarily to the set of CF nets.

Lemma 2 highlights strong similarities in the reachability graph between two sequences sharing the same starting state and the same destination state. The proof makes use of reverse sequences feasible in reverse WMGs.

Lemma 2 (Equivalent sequences in WMGs). *Let N be a connected WMG. Assume the existence of markings M, M_1 and sequences σ, σ' such that $M[\sigma\rangle M_1$ and $M[\sigma'\rangle M_1$. If N has no T-semiflow, then $\mathbf{P}(\sigma) = \mathbf{P}(\sigma')$. Otherwise, either $\mathbf{P}(\sigma) = \mathbf{P}(\sigma')$, or there exists an integer $k > 0$ such that $\mathbf{P}(\sigma) = \mathbf{P}(\sigma') + k.\pi$ or $\mathbf{P}(\sigma) + k.\pi = \mathbf{P}(\sigma')$, where π is the unique minimal T-semiflow of N.*

Proof. Let us assume that $\mathbf{P}(\sigma) \neq \mathbf{P}(\sigma')$. We show in the following that N has necessarily a T-semiflow in this case, proving the first claim by contraposition. Since N is a WMG, it is persistent. Defining $\tau = \sigma \stackrel{\bullet}{-} \sigma'$ and $\tau' = \sigma' \stackrel{\bullet}{-} \sigma$, applying Keller's theorem (Theorem 1), we have for some marking M_2 that $M_1[\tau\rangle M_2$ and $M_1[\tau'\rangle M_2$. By Lemma 1, τ and τ' have disjoint supports, $\tau \stackrel{\bullet}{-} \tau' = \tau$ and $\tau' \stackrel{\bullet}{-} \tau = \tau'$. Thus, applying Keller's theorem, a marking M_3 is reached from M_2 by firing τ or τ'. Iterating this process up to any positive integer i, some marking M_{i+1} is reached from M_i with $M_i[\tau\rangle M_{i+1}$ and $M_i[\tau'\rangle M_{i+1}$.

Now, in the reverse net $-N$, which is also a WMG, since $M_2[(\tau)^{-1}\rangle M_1$ and $M_2[(\tau')^{-1}\rangle M_1$, still with disjoint supports, we can construct markings M_0, M_{-1}, ... such that $\forall i \in \mathbb{Z}$, $M_i[(\tau)^{-1}\rangle M_{i-1}$ and $M_i[(\tau')^{-1}\rangle$ M_{i-1}, i.e. also $M_{i-1}[\tau\rangle M_i$ and $M_{i-1}[\tau'\rangle M_i$. If all M_i's are (pairwisely) different, this leads to a two-way uniform chain for the system (N, M_1), contradicting Proposition 1. Consequently, for some $i, j \in \mathbb{Z}$ with $i \neq j$, we have $M_i = M_j$, and since σ, σ' are different, they are not both empty and τ, τ' cannot be both empty. Thus, N has a T-semiflow, proving the first claim of the lemma.

For the second claim, either $\mathbf{P}(\sigma) = \mathbf{P}(\sigma')$ or $\mathbf{P}(\sigma) \neq \mathbf{P}(\sigma')$. Consider the latter case: from the first part of the proof, taking the same notation, there is a positive integer n such that τ^n and τ'^n are cycles appearing in the reachability graph of the system (N, M). Since the supports of τ and τ' are not both empty, Proposition 2 applies: there is a unique minimal T-semiflow π, whose support is

T, and integers $k, k' \geq 0$ exist such that $\mathbf{P}(\tau^n) = k \cdot \pi$ and $\mathbf{P}(\tau'^n) = k' \cdot \pi$, where $k > 0$ or $k' > 0$. Since the supports of τ and τ' are disjoint and the support of any cycle is T, then either $k' = 0$, $\tau' = \epsilon$, τ is a cycle and $\mathbf{P}(\sigma) \gneqq \mathbf{P}(\sigma')$, or $k = 0$, $\tau = \epsilon$, τ' is a cycle and $\mathbf{P}(\sigma') \gneqq \mathbf{P}(\sigma)$. Thus, either $\mathbf{P}(\sigma) = \mathbf{P}(\sigma') + \mathbf{P}(\tau) = \mathbf{P}(\sigma') + q.\pi$ for an integer $q > 0$ or $\mathbf{P}(\sigma') = \mathbf{P}(\sigma) + \mathbf{P}(\tau') = \mathbf{P}(\sigma) + q.\pi$ for an integer $q > 0$. Hence the claim. □

In [26], in the proof of Theorem 4.8, it is mentioned that each WMG is backward persistent without a proof, basing on the fact that the reverse of a WMG is still a WMG, hence a persistent net. However, this property needs to be proved carefully: since $M_1[a\rangle M$ and $M_2[b\rangle M$, Keller's theorem implies the existence of a marking M' reachable from $(-N, M_1)$ and $(-N, M_2)$, such that $M'[a\rangle M_2$ and $M'[b\rangle M_1$ in the original system; however, the reachability of M' in the original system, under the assumption of reachability for M_1 and M_2, is not obvious. In the following, we show it is indeed the case.

Theorem 2 (Backward persistence of WMGs). *In a connected WMG system $\zeta = (N, M_0)$, let us assume that markings M_1, M_2, M are reachable and that, for two different labels a and b, $M_1[a\rangle M$ and $M_2[b\rangle M$. Then, a marking M' is reachable in ζ such that $M'[a\rangle M_2$ and $M'[b\rangle M_1$.*

Proof. Let us write $N = (P, T, W)$ and introduce two sequences σ_1 and σ_2 enabled in ζ such that $M_0[\sigma_1\rangle M_1$ and $M_0[\sigma_2\rangle M_2$. From the previous remarks, we know that M' is reachable in the reverse system $(-N, M_1)$, hence belongs to $\mathbb{N}^P$. It remains to show that $M' \in [\zeta\rangle$.

From Lemma 2, either $\mathbf{P}(\sigma_1 a) = \mathbf{P}(\sigma_2 b)$ or, without loss of generality, $\mathbf{P}(\sigma_1 a) = \mathbf{P}(\sigma_2 b) + k \cdot \pi$, where π is the unique minimal T-semiflow of N, with support T. Then, in either case, b occurs at least once in σ_1. For the reverse net $-N$, we have $M_1[\sigma_1^{-1}\rangle M_0$, and from Keller's theorem, we have $M_1[b\rangle M'$ and $M_2[a\rangle M'$; we also have $M_0[b \stackrel{\bullet}{-} \sigma_1^{-1}\rangle M''$ and $M'[\sigma_1^{-1} \stackrel{\bullet}{-} b\rangle M''$. Since b occurs in σ_1, hence also in σ_1^{-1}, $b \stackrel{\bullet}{-} \sigma_1^{-1} = \epsilon$ and $M'' = M_0$. Going back to ζ, we deduce that $M_0[(\sigma_1^{-1} \stackrel{\bullet}{-} b)^{-1}\rangle M'$, thus M' is reachable in ζ. □

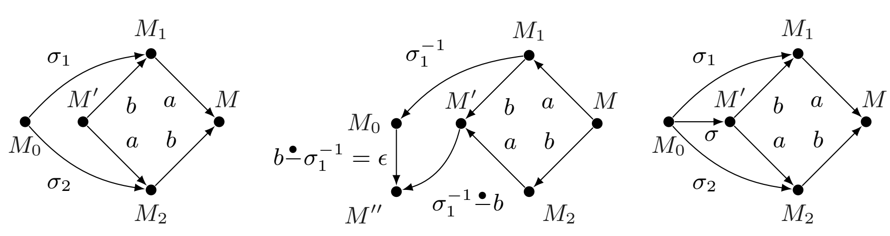


Fig. 3. Illustration of the proof of Theorem 2: the initial assumptions are depicted on the left, the sequences in the reverse system $(-N, M)$ are depicted in the middle, where the sequence leading to M'' from M_0 equals ϵ, implying that $M'' = M_0$. In the original system, we deduce the reachability of M' from M_0 on the right, with $\sigma = (\sigma_1^{-1} \stackrel{\bullet}{-} b)^{-1}$.

Theorem 2 becomes wrong for FA systems, thus also for CF systems. Indeed, a non-backward persistent FA system is provided in Figure 4.

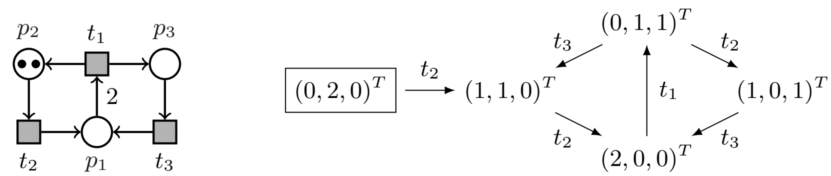


Fig. 4. A Fork-Attribution (FA) system on the left and its reachability graph on the right, where the initial marking is boxed. The FA system is not backward persistent, since the marking $(1,1,0)^T$ can be reached from two predecessors by firing t_2 and t_3 respectively from the initial marking and from $(0,1,1)^T$, but the initial marking has no predecessor.

3.3 Fireability of T-vectors in WMGs

In this subsection, we develop conditions for the existence of enabled sequences corresponding to given Parikh vectors. For that purpose, we borrow some vocabulary from [26, 28] as follows: for a net with incidence matrix C, we say that a marking M is *potentially reachable* from a marking M_0 if a T-vector ν exists such that $M = M_0 + C \cdot \nu$. If, additionally, a sequence σ is feasible in (N, M_0) such that $\mathbf{P}(\sigma) = \nu$, we say that ν is fireable (or feasible, or realisable) at M_0.

Lemma 3 (Realisable T-vectors in WMGs). *Let $N = (P, T, W)$ be a WMG with incidence matrix C. Let M be a marking and $\nu \in \mathbb{N}^T$ be a T-vector such that $M + C \cdot \nu \geq 0$. Let T_1 be the support of ν, $P_1 = {}^\bullet T_1 \cap T_1^\bullet$, σ' a transition sequence such that $\nu \leq \mathbf{P}(\sigma')$, and M' be a marking such that $\forall p \in P_1 : M'(p) = M(p)$. Then, if $M'[\sigma'\rangle$, there is a firing sequence $M[\sigma\rangle$ such that $\mathbf{P}(\sigma) = \nu$.*

Proof. By induction on the size of ν. If $\nu = \mathbb{0}$, the property is clearly true. Otherwise, let t be the first transition of T_1 occurring in σ', i.e. $\sigma' = \sigma_1' t \sigma_2'$ with $\nu(t_i) = 0$ for each t_i in σ_1'.

Assume that $\neg M[t\rangle$, then for some $p \in {}^\bullet t$, $M(p) < W(p,t)$. Since $M + C \cdot \nu \geq 0$, there is $t' \in {}^\bullet p$, $t' \neq t$, such that $t' \in T_1$, and t' is unique in ${}^\bullet p$ since N is a WMG. This contradicts the fact that $M'[\sigma_1' t\rangle$ since t' does not occur in σ_1'. Hence, we assume that $M[t\rangle M_1$ and $M'[\sigma_1'\rangle M''[t\rangle M_1'[\sigma_2'\rangle$.

Since the net is a WMG, the only transitions able to modify the places in P_1 are in T_1. Thus, we have $M(p) = M'(p) = M''(p)$ and $M_1(p) = M_1'(p)$ for each $p \in P_1$ (no transition of T_1 belongs to σ_1'). Let us denote by δ_t the T-vector with value 1 for t, 0 elsewhere.

Hence, the induction hypothesis applies to $\nu - \delta_t \leq \mathbf{P}(\sigma_2')$ from the markings M_1 and M_1', and there is a firing sequence $M_1[\sigma_1\rangle$ with $\mathbf{P}(\sigma_1) = \nu - \delta_t$. Thus, the sequence $\sigma = t\sigma_1$ with Parikh vector ν is enabled at M. The lemma results. $\square$

Instantiating Lemma 3 with $M = M'$, we deduce the next corollary.

Corollary 1 (Potential reachability in WMGs). *Let $N = (P, T, W)$ be a WMG with incidence matrix C. Let M be any marking and $\nu \in \mathbb{N}^T$ be a T-vector such that $M' = M + C \cdot \nu \geq 0$. Let σ be a transition sequence such that $\nu \leq \mathbf{P}(\sigma)$. Then, if $M[\sigma\rangle$, there is a firing sequence $M[\sigma'\rangle M'$ such that $\mathbf{P}(\sigma') = \nu$.*

Lemma 3 and Corollary 1 are not valid in the class of FA systems. Indeed, denoting by C the incidence matrix of the system in Figure 4, by M_0 its initial marking $(0,2,0)^T$ and by ν the T-vector $(1,1,1)^T$, we have: $M_0 = M_0 + C \cdot \nu$, and the sequence $\sigma = t_2 t_2 t_1 t_3$ is enabled at M_0, with $\mathbf{P}(\sigma) \geq \nu$. However, there is no initially feasible sequence whose Parikh vector equals ν.

We present next the notion of a *maximal execution vector* in order to obtain Theorem 3 on potentially reachable markings below.

Definition 3 (Maximal execution vector in WMGs). *Let ζ be a WMG system whose set of transitions is T. We denote by $maxex_\zeta : T \to \mathbb{N} \cup \{\infty\}$ the extended T-vector satisfying: $\forall t \in T$, $maxex_\zeta(t)$ is the maximal number of times t may be executed in firing sequences of ζ, allowing the case $maxex_\zeta(t) = \infty$.*

Theorem 3 (Potential reachability in WMGs, revised). *Let $\zeta = (N, M_0)$ be a WMG with incidence matrix C. Let $\nu \in \mathbb{N}^T$ be a T-vector such that $M = M_0 + C \cdot \nu \geq 0$. Let $maxex_\zeta$ be the maximal execution vector of ζ. Then, there exists a firing sequence $M_0[\sigma\rangle M$ with $\mathbf{P}(\sigma) = \nu$ if and only if $\nu \leq maxex_\zeta$.*

Proof. Suppose that $M_0[\sigma\rangle M$ with $\mathbf{P}(\sigma) = \nu$. Since σ can be fired at M_0, we deduce, from the definition of $maxex_\zeta$, that $\forall t \in T$, $maxex_\zeta(t) \geq \mathbf{P}(\sigma)(t)$, thus $\nu \leq maxex_\zeta$.

Conversely, suppose that $\nu \leq maxex_\zeta$. Then, for each $t \in T$, since $\nu(t) \leq maxex_\zeta(t)$, there is a finite firing sequence $M_0[\sigma_t\rangle$ such that $\nu(t) \leq \mathbf{P}(\sigma_t)(t)$. By persistence and Keller's theorem (applied $|T| - 1$ times), there is a finite firing sequence $M_0[\sigma'\rangle$ such that $\forall t \in T : \mathbf{P}(\sigma_t)(t) \leq \mathbf{P}(\sigma')(t)$, hence $\nu \leq \mathbf{P}(\sigma')$ and Corollary 1 applies. ☐

In this section, we delineated several properties on the reachability graph of WMGs. In the next section, we exploit some of these conditions, notably persistence and backward persistence, to synthesise a WMG from a given lts, when possible.

4 Synthesis of Connected, Bounded, Weakly Live WMGs

In the domain of Petri net synthesis from labelled transition systems, the aim is to build a Petri net system whose reachability graph is isomorphic to a given lts, when it exists. Usually, one has to check first some necessary structural properties of the lts. In some rare cases, such conditions have been proven sufficient for ensuring the existence of a solution (sometimes a unique minimal one) in the class considered and for driving the synthesis process [2, 4]. However, in most cases, the known synthesis methods need a combination of such necessary conditions with other computational checks and constructions [1, 3, 5, 7, 6, 9].

In this section, we focus on finite, totally reachable and *weakly live* lts, the latter property meaning that each label of T occurs at least once in the lts. We build a procedure synthesising a connected WMG solving such lts when possible.

First, in Subsection 4.1, we highlight necessary conditions of WMG-solvability, notably persistence, backward persistence and the existence of specific cycles. We also build a counter-example showing that these conditions, when satisfied, are not sufficient to ensure WMG-solvability.

Then, in Subsection 4.2, we highlight constraints induced by each place and we delineate two subsets of the lts states that are particularly relevant to WMG-synthesis. By focusing the analysis on these states, the number of checking steps is potentially reduced.

Finally, in Subsections 4.3 and 4.4, we define systems of constraints for two kinds of lts shapes: the cyclic case, i.e. when the lts is strongly connected (hence reversible), and the acyclic case, i.e. when the lts does not contain any cycle. We show these two cases to contain all the lts being solvable by a connected, bounded and weakly live WMG. Also, the number of constraints is reduced by checking only the relevant states defined in Subsection 4.2. When these systems have a solution, we obtain a WMG solving the lts. To extend this method to all the WMG-solvable lts, the decomposition technique developed in [14–16] to factorise a lts into *prime factors*, i.e. factors that cannot be further factorised and hence should correspond to connected nets, can finally be exploited.

4.1 Necessary Conditions for Solvability with Connected WMGs

For a synthesis into a connected WMG to succeed, the given lts must satisfy the conditions of Proposition 1, the properties described in Proposition 2, as well as persistence and backward persistence, as proved in Theorem 2. The boundedness of the WMG obtained stems from the finiteness of the lts. We capture part of these conditions with the next notation **b** and **c** and explain the relationship between the existence of a cycle in the lts and property **c**.

Properties b and c. For any lts $TS = (S, \to, T, \iota)$, we denote by:

- **b** the property: *TS* is finite, weakly periodic, deterministic and backward deterministic, persistent and backward persistent, totally reachable;
- **c** the property: *TS* is strongly connected, all its small cycles have the same prime Parikh vector π with support T, and $\mathbf{P}(\alpha)$ is a multiple of π for each cycle α.

Let us consider the case in which the finite lts contains a cycle. Then, the (finite) reachability graph of any connected and bounded WMG solving this lts contains a cycle. Thus, from Proposition 2, the cycle contains all transitions. By Corollary 4 in [27], the system is live, and by backward persistence, it is also reversible, implying the strong connectedness of the reachability graph. Consequently, we have to consider only two cases: the given lts is either acyclic or is strongly connected, the second case being considered in property **c**.

Without loss of generality, we assume in the sequel that the lts considered are weakly live. The next lemma presents relationships between properties relevant to the synthesis.

Lemma 4 (Determinism, reversibility, cycle consistence). *Let us consider a weakly live lts $TS = (S, \rightarrow, T, \iota)$.*
1) *If TS satisfies* **b***, it also satisfies the full determinism and full backward determinism.*
2) *If TS satisfies* **b** *and is acyclic, all the sequences between any two states have the same Parikh vector.*
3) *If TS satisfies* **b** *and contains a small prime cycle with support T and Parikh vector π, then TS satisfies property* **c***, there is a small prime cycle around each state, each arc belongs to a small prime cycle, TS satisfies the strong cycle consistence; also, for any two states s_1 and s_2, there is a sequence from s_1 to s_2 whose Parikh vector δ is not greater than nor equal to π, and each other sequence σ from s_1 to s_2 satisfies $\mathbf{P}(\sigma) = \mathbf{P}(\delta) + k \cdot \pi$ for a non-negative integer k.*

Proof. 1) Full determinism and full backward determinism arise directly from determinism and backward determinism and from persistence and backward persistence, with the aid of Proposition 3 applied to TS and to its reverse version.

2) If the lts is acyclic, satisfies **b** and, for some $s \in S$ and $s' \in [s\rangle$, we have $s[\alpha\rangle s'$ as well as $s[\beta\rangle s'$ with $\mathbf{P}(\alpha) \neq \mathbf{P}(\beta)$, and $\alpha \dot{-} \beta$ or $\beta \dot{-} \alpha$ is non-empty (both of them may be non-empty). Then, as in the proof of Lemma 2, with the aid of Keller's theorem and of Lemma 1, we can build a uniform chain $s'[\alpha \dot{-} \beta\rangle s_1[\alpha \dot{-} \beta\rangle s_2 \cdots$ and $s'[\beta \dot{-} \alpha\rangle s_1[\beta \dot{-} \alpha\rangle s_2 \cdots$. Since the lts is finite, there must exist positive integers i and j such that $i < j$ and $s_i = s_j$, forming a non-empty cycle, hence a contradiction with the acyclicity.

3) In the rest of the proof, we suppose that the lts satisfies **b** and contains a small prime cycle α around some state $s \in S$ with support T. Determinism and persistence imply that cycles can be pushed forward Parikh-equivalently, i.e.: if $s[\alpha\rangle s \wedge s[t\rangle s'$, then $s'[\alpha'\rangle s'$ for some α' with $\mathbf{P}(\alpha') = \mathbf{P}(\alpha)$ (applying Keller's theorem). Symmetrically, backward determinism and backward persistence imply that cycles can be pushed backward Parikh-equivalently.

Now, consider any non-empty cycle β around some state s' in TS. Both cycles α and β can be pushed backward Parikh-equivalently to the initial state ι (since s and s' are reachable from ι by total reachability). Using Keller's theorem, both support-disjoint sequences $\alpha \dot{-} \beta$ and $\beta \dot{-} \alpha$ are feasible at ι and lead to some marking s_0. Since $(\alpha \dot{-} \beta)^n$ and $(\beta \dot{-} \alpha)^n$ are feasible at ι for every positive integer n while the lts is finite, there exists a positive integer m such that $(\alpha \dot{-} \beta)^m$ and $(\beta \dot{-} \alpha)^m$ are cycles. Since the lts is also weakly periodic, deterministic and backward deterministic, both $\alpha \dot{-} \beta$ and $\beta \dot{-} \alpha$ are cycles. Since $\alpha, \beta \neq \epsilon$ and $supp(\alpha) = T$, we have $\mathbf{P}(\alpha \dot{-} \beta) \lneq \mathbf{P}(\alpha)$. Hence, if $\alpha \dot{-} \beta \neq \epsilon$, it forms a smaller cycle, contradicting the fact that α is already a small cycle. Thus, necessarily, $\alpha \dot{-} \beta = \epsilon$, implying that $\mathbf{P}(\beta) \geq \mathbf{P}(\alpha)$. Suppose that $\mathbf{P}(\beta)$ is not a multiple of $\mathbf{P}(\alpha)$. Denote by k the largest integer such that $\mathbf{P}(\beta) \geq k \cdot \mathbf{P}(\alpha)$ and $\beta' =$

$\beta \stackrel{\bullet}{-} \alpha^k \neq \epsilon$. Necessarily, $\mathbf{P}(\alpha) \not\geq \mathbf{P}(\beta')$ and $\mathbf{P}(\beta') \not\geq \mathbf{P}(\alpha)$, implying that $\mathbf{P}(\alpha) \gneqq \mathbf{P}(\alpha \stackrel{\bullet}{-} \beta') \gneqq \mathbb{0}$, where $\alpha \stackrel{\bullet}{-} \beta'$ is a cycle, contradicting the fact that α is a small cycle. We deduce that $\mathbf{P}(\beta)$, as well as each other Parikh vector of each non-empty cycle of the lts, is a multiple of $\mathbf{P}(\alpha) = \pi$.

Hence, from total reachability and persistence, there is a small prime cycle (with Parikh vector π) around the initial state, as well as around any state.

Since there is a small cycle with support T around each state, by Keller's theorem each arc can be extended into a cycle: $s[t\rangle s'$ implies there is a sequence $s'[\gamma\rangle s$ with $\mathbf{P}(t\gamma) = \pi$. As a consequence, TS is reversible, thus strongly connected.

For any cycle $s[\beta\rangle s$, from the above, we have that $\mathbf{P}(\beta) = k \cdot \pi$ for some integer $k \geq 0$. Now, if a sequence $s[\gamma\rangle s'$ is such that $k_1 \cdot \mathbf{P}(\gamma) = k_2 \cdot \pi$ for some positive integers k_1, k_2, since π is prime k_1 must divide k_2; let us denote by k' the integer k_2/k_1. We have $\mathbf{P}(\gamma) = k' \cdot \pi$ so that by full determinism $s = s'$, hence the strong cycle consistence.

Consider a sequence $s[\alpha\rangle s'$ in TS. Suppose that $\mathbf{P}(\alpha) \geq \pi$. Since there is a small cycle γ with Parikh vector π around s, we build a shorter sequence by applying Keller's theorem as follows: $\alpha \stackrel{\bullet}{-} \gamma$ is fireable at s and leads to s'. We can build such shorter sequences until we get a sequence $s[\alpha'\rangle s'$ with $\mathbf{P}(\alpha') \not\geq \pi$ and $\mathbf{P}(\alpha) = \mathbf{P}(\alpha') + k \cdot \pi$ for some integer $k > 0$. If we start from another sequence $s[\beta\rangle s'$, we get similarly $s[\beta'\rangle s'$ with $\mathbf{P}(\beta') \not\geq \pi$, and $\mathbf{P}(\beta) = \mathbf{P}(\beta') + h \cdot \pi$ for a non-negative integer h. If $\mathbf{P}(\beta') \neq \mathbf{P}(\alpha')$, then we have two cases.

In the first case, one of them is greater than the other one, without loss of generality $\mathbf{P}(\beta') \geq \mathbf{P}(\alpha')$, in which case, with Keller's theorem, we can construct from s' the cycle $\beta' \stackrel{\bullet}{-} \alpha'$ with $\mathbf{P}(\beta' \stackrel{\bullet}{-} \alpha') \not\geq \mathbf{P}(\pi)$ a contradiction with the fact that the Parikh vector of every cycle of TS is a multiple of π (with support T).

In the second case, $\beta' \stackrel{\bullet}{-} \alpha'$ and $\alpha' \stackrel{\bullet}{-} \beta'$ are both non-empty with disjoint supports, their support containing at least one null component. In this second case, we construct from $s_1 = s'$ a chain $s_1[\sigma\rangle s_2[\sigma\rangle s_3 \cdots$, with $\sigma = \beta' \stackrel{\bullet}{-} \alpha'$ as well as for $\sigma = \alpha' \stackrel{\bullet}{-} \beta'$. Since the lts is finite, we must have $s_i = s_j$ for some $i < j$, hence a cycle with Parikh vector $n \cdot \pi = (j - i) \cdot \mathbf{P}(\sigma)$ for some positive integer n, which is incompatible with the fact that the support of π is T and the support of σ does not contain all transitions.

Thus, we get a contradiction in both cases, implying that $\mathbf{P}(\beta') = \mathbf{P}(\alpha')$. We deduce that for every sequence σ from s_1 to s_2, there exists a sequence $\delta \not\geq \pi$ from s_1 to s_2 such that $\mathbf{P}(\sigma) = \mathbf{P}(\delta) + k \cdot \pi$ for some non-negative integer k. □

Insufficiency of the necessary conditions b and c for WMG-solvability. In Figure 5, we provide an example of an FA system whose reachability graph satisfies all conditions of properties **b** and **c** but is not WMG-solvable. We deduce that these conditions, when satisfied by a given lts, are not sufficient for ensuring the existence of a solution in the WMG subclass. Indeed, in Figure 5, each possible attempt of a construction leads to a contradiction, as detailed next.

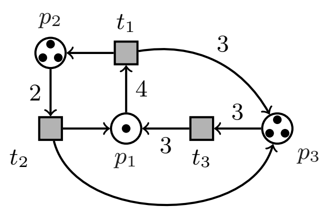
- Non-existence of a WMG solution with six places:
  To obtain a WMG solution structured as on the right of Figure 5, the

following sequences must be feasible at the initial marking M_0:

$$t_3t_1 \Rightarrow M_0(p_1) \geq 3 \qquad t_3t_1t_3t_2t_1t_3t_2t_1t_3t_2t_3 \Rightarrow M_0(p_4) \geq 9$$
$$t_3t_1t_2t_3t_1t_3t_2t_1 \Rightarrow M_0(p_2) \geq 3 \qquad t_3t_1t_3 \Rightarrow M_0(p_5) \geq 6$$
$$t_2t_3t_1t_2 \Rightarrow M_0(p_3) \geq 9 \qquad t_2t_3t_1t_2t_3t_1t_3t_1t_2 \Rightarrow M_0(p_6) \geq 12$$

The sequence $\sigma = t_2t_3t_1t_3t_1t_3t_2t_1t_3t_3$ is then feasible in such a constrained WMG but is not enabled in the FA system ζ.

– Non-existence of a WMG solution with fewer places:
Since the previous WMG system with six places and its necessary marking are too permissive, we deduce that the same contradicting sequence σ is also feasible in all less constrained WMGs, typically obtained by removing some places while retaining the necessary initial marking in the other places.



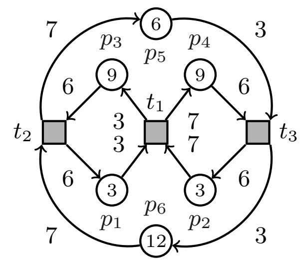


Fig. 5. A Fork-Attribution (FA) system ζ is pictured on the left. Its minimal prime T-semiflow $\pi = (6, 3, 7)$ equals the Parikh vector of each small cycle of $RG(\zeta)$. The latter is persistent and backward persistent, reversible, finite and fulfills properties **b** and **c**. The most constrained WMG solution ζ' whose reachability graph could be isomorphic to $RG(\zeta)$ is depicted on the right: its weights are directly deduced from π and, in each place, the given amount of tokens is necessary to enable the sequences of $RG(\zeta)$. However, this necessary initial marking already enables a sequence that is not feasible in ζ, namely $\sigma = t_2t_3t_1t_3t_1t_3t_2t_1t_3t_3$. Since every possible variant ζ'' of ζ' is less constrained than ζ', each such ζ'' also enables σ. We deduce that no WMG solves $RG(\zeta)$.

Checking the necessary conditions in a pre-synthesis phase. Since the lts is finite, all the necessary properties for solvability can be checked in a naive way. However, some algorithmic improvement can be achieved by considering an adequate checking order. For instance, from Proposition 3, property **b** implies full determinism and full backward determinism, whose checking is thus avoided. In the next subsection, we exhibit subsets of states of the lts whose analysis is sufficient to ensure some constraints, allowing to perform fewer operations.

4.2 Constraints and Subsets of States Relevant to WMG-Synthesis

In the following, we describe some constraints that must be fulfilled in order to synthesise a WMG. Also, we define two subsets of the states of the given lts that are sufficient to check in order to fulfill several constraints over all states, decreasing potentially the size of the systems of constraints to solve.

A WMG synthesis amounts to build places of the kind schematised in Figure 6.

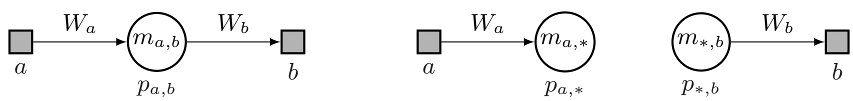


Fig. 6. Possible types of places for the synthesis of a WMG (N, M_0), with initial marking $m_{a,b} = M_0(p_{a,b})$, $m_{a,*} = M_0(p_{a,*})$ and $m_{*,b} = M_0(p_{*,b})$.

Constraints related to places in the WMG. Note that a place $p_{a,*}$ is equivalent to a place $p_{a,b}$ with $W_b = 0$, and a place $p_{*,b}$ is equivalent to a place $p_{a,b}$ with $W_a = 0$. In a place $p_{a,b}$, we can always choose W_a and W_b relatively prime without loss of generality, with an adequate initial marking M_0. If a and b are the same label, then we have a single transition and the place is equivalent to either a place $p_{a,*}$, $p_{*,a}$ or no place at all, depending on the sign of the difference between W_a and W_b. In a place $p_{a,*}$, the initial marking $M_0(p_{a,*})$ may always be chosen as 0 and the weight W_a as 1. In a place $p_{*,b}$, we must have $W_b \leq M_0(p_{*,b})$ (otherwise the lts would not be weakly live), and the weight W_b can always be chosen as 1, with an adequate choice of the initial marking M_0.

If $T = \emptyset$, TS is reduced to its initial state and the (minimal) solution is the empty Petri net. If $T = \{a\}$ is a singleton, either TS is acyclic, in the form of a single chain, and the minimal solution is a place $p_{*,a}$, with an initial marking deduced from the length of the chain, or it is a loop $\iota[a\rangle\iota$ with a minimal solution reduced to a transition a without any place. Hence, in the following, we assume without loss of generality that $|T| > 1$. We shall also assume that the lts to be synthesised satisfies property **b** and either acyclicity or **c**.

M_0 is the marking corresponding to the initial state ι; consider any state $s \in S$ with a shortest sequence from ι to s, meaning that no other sequence from ι to s has a smaller Parikh vector. By Lemma 4 (point 2 or 3), such a sequence exists, and all such sequences from ι to s share the same Parikh vector Δ_s. The marking corresponding to state s is given by $M_s(p_{a,b}) = M_0(p_{a,b}) + \Delta_s(a) \cdot W_a - \Delta_s(b) \cdot W_b$. The next conditions are necessary and sufficient for allowing (and realising) a synthesis, and are related to the classical regional approach [1]:

- The number of tokens in $p_{a,b}$ must remain non-negative at each reachable marking described by a state in S.
- For each state s not allowing b, there must exist a place p such that W_b is larger than $M_s(p)$, where M_s is the marking associated to s.
- Any two different states s', s'' must be distinguished by a place p' such that $M_{s'}(p') \neq M_{s''}(p')$.

In many cases, notably for CF and MG synthesis [3, 4, 9, 12], hence in this study, the last constraint, called the *separation property*, arises from the other two and from the assumptions on TS.

Two subsets of states relevant to the WMG-synthesis. The first two constraints above are linked to two particular subsets of states of TS: for each label $x \in T$, we define

$$OX(x) = \{r \in S \mid r[t\rangle \Rightarrow t = x\} \text{ and } NXX(x) = \{s \in S | \neg s[x\rangle \wedge \forall s' \in s^\bullet : s'[x\rangle\}.$$

For each state s in $OX(x)$ (the notation stemming from "Only X"), the only arc starting at s, if any, is labelled x. Let us consider a place $p_{a,b}$ and a longest sequence without a starting from some state s. This sequence is finite since the lts is finite, and each cycle along the sequence, if any, has support T, hence contains an a. Thus, we reach a state r either without successor (this may only occur if the lts is acyclic) or with a single output a, hence in $OX(a)$ in both cases, and $M_s(p_{a,b}) \geq M_r(p_{a,b})$. As a consequence, to check that all markings of $p_{a,b}$ reachable from ι are non-negative, we only have to check the states in $OX(a)$: $r \in OX(a) \Rightarrow M_r(p_{a,b}) \geq 0$ and $M_s(p_{a,b}) \geq 0$.

For each state s in $NXX(x)$, x cannot be executed at s (hence the first two letters NX of the notation), but in each next state s', if any, x is enabled (hence the last letter X of the notation). Let us assume that a place $p_{a,b}$ may be used to exclude performing b at some state s (i.e. $\neg s[b\rangle$), meaning $M_s(p_{a,b}) < W_b$. If $s'[t\rangle s$ with $\neg s'[b\rangle$ (which implies $t \neq b$), then $M_s(p_{a,b}) \geq M_{s'}(p_{a,b})$, so that the same place $p_{a,b}$ disables b at s'. Moreover, the longest chains of states excluding to perform b are necessarily finite since b occurs in any non-empty cycle; hence they all end in states of $NXX(b)$. As a consequence, in order to exclude performing b when necessary, one only has to find, for each state $r \in NXX(b)$, a place $p_{a,b}$ such that $M_r(p_{a,b}) < W_b$ (while allowing all valid transitions, as expressed through $OX(a)$). In some cases, a same place $p_{a,b}$ can be used for several states in $NXX(b)$.

In our case, for any label x, the states in $NXX(x)$ have a very special shape, highlighted in the next lemma whose proof is illustrated in Figure 7.

Lemma 5 (Single outputs of the states in NXX**).** *Let* $TS = (S, \rightarrow, T, \iota)$ *be a lts satisfying property* **b**. *If* $x, a, b \in T$, $r \in NXX(x)$, $r[a\rangle$ *and* $r[b\rangle$, *then* $a = b$.

Proof. Let us assume that $r[a\rangle r_1$ and $r[b\rangle s_1$ with $a \neq b$. Since $r \in NXX(x)$, we have $r_1[x\rangle r_2$ and $s_1[x\rangle s_2$ for some states r_2, s_2. By persistence (and determinism), we also have $r_1[b\rangle s$, $s_1[a\rangle s$, $s[x\rangle s'$, $r_2[b\rangle s'$ and $s_2[a\rangle s'$ for some s, s'. By backward persistence, we then have $s''[a\rangle r_2$ and $s''[b\rangle s_2$ for some s'', as well as $s'''[x\rangle s''$ and $s'''[b\rangle s_1$ for some s'''. Finally, by backward determinism, $s''' = r$ and $r[x\rangle$, contradicting the fact that $r \in NXX(x)$. $\square$

If TS is acyclic, by persistence there is a unique (maximal) state without successor; let us call it s_∞; we then have $s_\infty \in \cap_{x \in T} NXX(x)$. If TS is cyclic, there is no such state. In any case, we may have several states s and label $a \neq x$ in some $NXX(x)$ with $s[ax\rangle$.

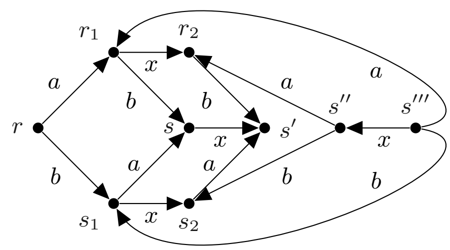


Fig. 7. Illustration of the proof of Lemma 5.

4.3 Computational Synthesis in the General Cyclic Case

Let $TS = (S, \rightarrow, T, \iota)$ be a lts satisfying properties **b** and **c**, denoting by π the unique minimal Parikh vector of small cycles, with support T. Each place $p_{a,b}$ must satisfy $W_a \cdot \pi(a) = W_b \cdot \pi(b)$, thus we can choose $W_a = \pi(b)$ and $W_b = \pi(a)$ (or any proportional values[3], in particular $\pi(b)/\gcd(\pi(a), \pi(b))$ and $\pi(a)/\gcd(\pi(a), \pi(b))$), and the only parameter that still needs to be fixed is the initial marking (plus the exact pairs a, b for which we need those places).

For each $b \in T$, we need such a place $p_{a,b}$ if there is a state $s \in NXX(b)$ such that $s[ab\rangle$ (otherwise, there is no way to enable a b after an a when b is not directly enabled). We denote by $pred(b)$ the set $\{a \in T | \exists s \in NXX(b), s[ab\rangle\}$ and, for any $a \in pred(b)$, $NXX(a, b) = \{s \in NXX(b) | s[ab\rangle\}$.

For each $a, b \in T$ such that $a \in pred(b)$, since $W_a = \pi(b)$ and $W_b = \pi(a)$, we have to solve the following constraints (in M_0, over the non-negative integers):

$$\begin{cases} \forall s \in OX(a) : M_0(p_{a,b}) \geq \Delta_s(b) \cdot \pi(a) - \Delta_s(a) \cdot \pi(b) \\ \forall s \in NXX(a, b) : M_0(p_{a,b}) < \Delta_s(b) \cdot \pi(a) - \Delta_s(a) \cdot \pi(b) + \pi(a) \end{cases}$$

This amounts to first compute

$$M_0(p_{a,b}) = \max_{s \in OX(a)} \{\Delta_s(b) \cdot \pi(a) - \Delta_s(a) \cdot \pi(b)\}$$

and then to check that, for each $s \in NXX(a, b)$,

$$M_0(p_{a,b}) < \Delta_s(b) \cdot \pi(a) - \Delta_s(a) \cdot \pi(b) + \pi(a).$$

If each such system of constraints is solvable, we obtain a WMG solution of TS. Otherwise, there is no solution and the reason is known.

4.4 Computational Synthesis in the General Acyclic Case

In the acyclic case, we may first apply the factorisation techniques of [14–16] to check if the given lts is prime and thus has a chance to have a connected

[3] This is the only way to define an adequate $p_{a,b}$; in particular, there is no $p_{*,b}$ or $p_{a,*}$.

solution. The weights W_a and W_b around the place $p_{a,b}$ are not constrained by a T-semiflow. Thus, we may need variants of such places (differing by the weights W_a, W_b and the initial marking). We may also need places $p_{*,b}$ and $p_{a,*}$; in particular, a place $p_{*,b}$ with $W_b = 1$ and $M_0 = \Delta_{s_\infty}(b)$ excludes executing b at the final state s_∞. Such a place may be redundant with other ones, but we do not aim here at building an optimal solution: we focus on the existence of a solution and on its construction.

In this acyclic case, the enabledness of labels is described by the first set of constraints below, using again the sufficient condition stating that the markings at states from $OX(a)$ must be non-negative. The last constraint expresses that the place is useful for excluding some transition from some state.

For each $b \in T$, $a \in pred(b)$ and $s \in NXX(a,b)$, we have to solve the following constraints (in $M_0(p_{a,b}), W_a, W_b \in \mathbb{N}$):

$$\begin{cases} \forall s' \in OX(a) : M_0(p_{a,b}) \geq \Delta_{s'}(b) \cdot W_b - \Delta_{s'}(a) \cdot W_a \\ M_0(p_{a,b}) < \Delta_s(b) \cdot W_b - \Delta_s(a) \cdot W_a + W_b. \end{cases}$$

To solve such a system, we can first consider the system in W_a and W_b:

$$\forall s' \in OX(a) : \Delta_s(b) \cdot W_b - \Delta_s(a) \cdot W_a + W_b > \Delta_{s'}(b) \cdot W_b - \Delta_{s'}(a) \cdot W_a$$

i.e.: $\forall s' \in OX(a) : [\Delta_s(b) - \Delta_{s'}(b) + 1] \cdot W_b > [\Delta_s(a) - \Delta_{s'}(a)] \cdot W_a$

and then check if there exists a solution satisfying:

$$\Delta_s(b) \cdot W_b - \Delta_s(a) \cdot W_a + W_b > 0.$$

If each such system of constraints is solvable, we obtain a WMG solution of TS. Otherwise, no solution exists and we know the reason.

5 Conclusions and Perspectives

Weighted marked graphs (WMGs) form a well-known subclass of Petri nets with numerous real-life applications. These nets have been extensively studied in previous works, leading to strong theoretical results.

For this class, we obtained new structural and behavioural properties, such as backward persistence. We also delineated necessary conditions that must be fulfilled by a labelled transition system to be WMG-solvable. We showed that these necessary conditions are not sufficient. Finally, we specialised the synthesis procedures devised for choice-free nets in [3, 5, 7, 6, 9] to WMG nets.

A perspective is to develop additional properties of WMGs in order to enhance the pre-synthesis phase, allowing to discard non-solvable systems promptly. Ideally, such properties would characterise the WMG-solvable labelled transition systems in a purely structural way, in the spirit of the methods designed for plain marked graphs and T-systems in [2, 4].

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