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On Liveness and Reversibility of Equal-Conflict Petri Nets

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Abstract. Weighted Petri nets provide convenient models of many man-made systems. Real applications are often required to possess the fundamental Petri net properties of liveness and reversibility, as liveness preserves all the functionalities (fireability of all transitions) of the system and reversibility lets the system return to its initial state (marking) using only internal operations.

Characterizations of both behavioral properties, liveness and reversibility, are known for well-formed weighted Choice-Free and ordinary Free-Choice Petri nets, which are special cases of Equal-Conflict Petri nets. However, reversibility is not well understood for this larger class, where choices must share equivalent preconditions, although characterizations of liveness are known.

In this paper, we provide the first characterization of reversibility for all live Equal-Conflict Petri nets by extending, in a weaker form, a known condition that applies to the Choice-Free and Free-Choice subclasses. We deduce the monotonicity of reversibility in the live Equal-Conflict class. We also give counter-examples for other classes where the characterization does not hold. Finally, we focus on well-formed Equal-Conflict Petri nets, for which we offer the first polynomial sufficient conditions for liveness and reversibility, contrasting with the previous exponential time conditions.

Keywords: Reversibility, liveness, Equal-Conflict, Petri net, characterization, polynomial time sufficient condition, polynomial markings.

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1. Introduction

1.1. Models, properties and applications

Petri nets constitute a highly expressive and intuitive operational model of discrete event systems, capturing the all-important mechanisms of synchronization, conflict and concurrency. Many of their fundamental behavioral properties are decidable, including liveness, boundedness and reachability of a marking [1]. Consequently, they have been fruitfully employed to model and analyze numerous real-life systems, such as embedded systems and digital hardware, flexible manufacturing systems, communication protocols and biological networks.

Importance of weights. In this paper, we study weighted Petri nets, a generalization well suited to the modeling of a variety of man-made and natural systems:

In the domain of embedded systems, Synchronous Data Flow (SDF) graphs [2] were introduced to model the communication between a finite set of concurrent and periodic processes on parallel architectures. In these graphs, used notably in the design of Digital Signal Processing (DSP) applications [3, 4, 5], the fixed weights represent the quantities of data conveyed. SDF graphs can be represented by T-systems, a weighted subclass of Petri nets [6].

Petri nets used to model flexible manufacturing systems (FMS) [7, 8] have weights that represent the bulk consumption and production of resources.

In the Petri net models employed for the computational analysis of biochemical systems, weights account for the stoichiometry of biochemical reactions [9].

Thus, in general, weights in Petri nets allow a compact representation of the volumes of data or resources exchanged.

Desired behavior of applications. Liveness and reversibility are two behavioral properties of Petri nets of great practical relevance. Liveness guarantees the possibility to enable within a finite number of steps any transition from any reachable marking. Reversibility allows to reach the initial state from every other reachable state, by means of only internal operations. Reversible systems benefit from a cyclic, regular, behavior from the start, thus avoiding a costly transient phase. In addition, reversibility permits error recovery and often greatly simplifies the study of the reachability graph.

The boundedness property is also essential for numerous applications. It asserts the existence of an upper bound on the maximal number of tokens in the system for all reachable markings (states), thus ensuring the finiteness of the reachability graph of the system.

As embedded systems must preserve all their functionalities over time, operate within bounded memory and exhibit a regular behavior, they constitute typical examples of applications that need to be live, bounded and reversible.

Difficulty of analysis and classical approaches. The behavioral properties of liveness, reversibility and boundedness, although decidable [1, 10, 11, 12], induce a high analysis complexity: the liveness and boundedness checking problems are EXPSPACE-hard [1, 13, 14], while the reversibility checking problem is PSPACE-hard [12]. Also, they do not imply one another in most weighted Petri nets [15]. Two common approaches are employed to alleviate these difficulties: limit the analysis to particular subclasses of Petri nets and study the underlying structure of the net to gain insight on the behavior. These approaches are frequently combined to improve the efficiency of the checking conditions.

Well-known subclasses of weighted Petri nets. Over the years, numerous weighted subclasses of Petri nets have been introduced and studied. Some important ones are defined by simple structural restrictions:

Ordinary Free-Choice systems [16, 17] force all output transitions of a place to share the same input places. These systems only have unit weights, in contrast with the following ones.

T-systems [6], in which each place has at most one input and one output, are strictly included in the weighted Choice-Free class [8], also known as *output non-branching* [18], in which each place has at most one output. Join-Free systems [8, 19] force every transition to have at most one input, hence forbid synchronizations. The intersection of the Choice-Free and Join-Free classes defines the Fork-Attribution systems [8].

Equal-Conflict systems [19, 20], also called homogeneous (extended) Free-Choice nets, are weighted and homogeneous, meaning that for each place p, all output weights of p are equal. Moreover, the Equal-Conflict class forces all output transitions of a place to share the same input places. Hence, this weighted class restricts the choices to equal firing preconditions and generalizes the weighted Choice-Free and ordinary Free-Choice classes.

Several structural generalizations of these classes have been studied, such as ordered Petri nets [21].

Relationship between the structure and the behavior. The well-formedness property states the existence of a live initialization (structural liveness) and ensures boundedness for each initialization (structural boundedness). Polynomial time methods have been developed to check well-formedness for several weighted subclasses of Petri nets, including the Join-Free and Equal-Conflict classes [19, 21, 22, 23, 24, 25, 26]. When a net is known to be well-formed, the ensuing challenge is to efficiently construct a live marking for this net. However, another difficulty arises as live markings are not always reversible [15].

For several classes of nets that are defined by particular structural properties, some behavioral equivalences related to liveness are known [21]. A necessary condition of structural liveness, based on the concept of *controllable siphons*, has been provided for homogeneous nets in [27], where (non-)preservation of liveness upon some increase of a live marking is also studied from the structural point of view.

Previous results on the behavior. For bounded Equal-Conflict systems, there exist dedicated characterizations of liveness based on integer linear programming and graph decomposition techniques [19, 20]. Equal-Conflict systems belong to the more general class of *ordered Petri nets*, for which a characterization of liveness in terms of *controlled siphons* has been developed in [21]. However, these methods do not induce polynomial time algorithms to check or ensure liveness in the Equal-Conflict class.

The existence of reachable home markings, which are defined as being reachable from every reachable marking, is stated for live well-formed Equal-Conflict systems in [20].

A polynomial-time necessary and sufficient condition for both liveness and reversibility has been found for the class of well-formed ordinary (unit-weighted) Free-Choice systems, a proper subclass of the Equal-Conflict systems [16, 17]. For bounded ordinary Free-Choice nets, liveness and structural liveness are known to be decidable in polynomial time [17, 22, 26].

Markings with a polynomial number of tokens that are built in polynomial time, called *polynomial markings*, have been presented for some other classes, thus inducing polynomial time sufficient conditions of liveness or reversibility. For the class of well-formed Join-Free nets, polynomial live markings were proposed in [28], with polynomial live reversible markings in the well-formed Fork-

Attribution subclass. In the case of well-formed Choice-Free nets, characterizations of reversibility have been found [8, 29] and polynomial live reversible markings were provided in [29] together with a polynomial time sufficient condition ensuring or checking both liveness and reversibility. These results extend conditions that were presented for T-systems and SDF graphs [30].

Main objectives. We aim at unifying and generalizing polynomial sufficient conditions of liveness, boundedness and reversibility obtained, notably, for T-systems and SDF graphs. The intention is to be able, at the design phase, to ensure efficiently that applications with models belonging to larger subclasses of weighted Petri nets possess these desirable behavioral properties. Since buffers in T-systems may have at most one writer and one reader, the production and consumption of data items present a form of determinism. Such predictability in the execution of the operations seriously limits the expressive power of T-systems. By introducing choices in the models some indeterminism is injected that augments their flexibility, thus permitting the analysis and design of more sophisticated systems. For that purpose, we focus in this study on the Equal-Conflict class which authorizes jointly, albeit in a restricted fashion, synchronizations, choices and weights. As seen above, several steps towards this objective have been performed in previous studies, notably for Equal-Conflict Petri nets and their subclasses. However, to our knowledge, non-trivial characterizations for both liveness and reversibility, with associated polynomial checking conditions, have not yet been found for the Equal-Conflict class.

1.2. Contributions

We first show that a previous characterization of liveness and reversibility by decomposition, developed for well-formed Choice-Free nets, cannot be extended to Equal-Conflict nets.

The central contribution of this study is the first non-trivial characterization of reversibility for all live Equal-Conflict systems. This result generalizes, in a weaker form, a condition that was originally developed for well-formed Choice-Free and ordinary Free-Choice systems. Our characterization is based on the notion of a *T-sequence*, specifically a firing sequence that contains every transition of the system and returns to the initial marking. More precisely, we prove that the existence of a feasible T-sequence is necessary and sufficient to ensure reversibility in live Equal-Conflict Petri nets, which are not necessarily bounded nor strongly connected.

We then use our new characterization to prove the monotonicity of reversibility in the live Equal-Conflict class, meaning the preservation of reversibility upon any increase of the live marking considered.

We also provide counter-examples showing that the characterization does not carry over to some larger classes.

Finally, we focus on the well-formed Equal-Conflict nets, for which we construct the first polynomial live reversible markings. By *polynomial*, we mean not only computed in polynomial time but also with a polynomial number of initial tokens. Besides, by the monotonicity property, these markings induce the first polynomial time sufficient conditions for liveness and reversibility in this class.

Comparing with [31], we introduce a new counter-example showing that a previous decomposition condition for Choice-Free systems does not propagate to larger classes. We also supply the result on the monotonicity of reversibility and provide further counter-examples showing failures of the characterization in other classes. Finally, we contribute the first polynomial time sufficient conditions for liveness and reversibility in well-formed Equal-Conflict Petri nets.

1.3. Organization of the paper

This paper is structured as follows. In Section 2, we formalize general definitions and notations concerning weighted Petri nets, including the subclasses and properties studied in this paper. We also recall some related properties and results.

In Section 3, we study how reversibility interacts with liveness, decompositions of well-formed nets and *T-sequences*, which are introduced on this occasion.

The central result, the characterization of reversibility for live Equal-Conflict systems, is presented in Section 4. We show that for these systems reversibility satisfies a monotonicity property. We also provide some insights on the complexity of reversibility checking in this class.

Counter-examples are provided in Section 5, showing that the characterization does not work in several other classes.

Section 6 focuses on well-formed Equal-Conflict Petri nets, for which we construct the first polynomial time sufficient conditions of liveness and reversibility.

Finally, Section 7 presents our conclusion with perspectives.

2. Classical definitions, notations and properties

In this section, we recall established definitions and notations related to the structure and behavior of Petri nets. First, we focus on nets, markings, systems, firing sequences and reachability graphs. Second, we formalize several fundamental notions, including liveness, reversibility, well-formedness and some algebraic properties. Third, we present the subclasses studied in this paper, including Join-Free and Equal-Conflict Petri nets. We also recall a characterization of well-formedness for the Choice-Free and Join-Free subclasses. Last, we present subsequences, subnets, components and some related notions.

2.1. Weighted and ordinary nets

A *(weighted) net* is a triple $N = (P, T, W)$ where:

- the sets P and T are finite and disjoint, their elements are respectively called places and transitions,
- $W : (P \times T) \cup (T \times P) \mapsto \mathbb{N}$ is a weight function.

$P \cup T$ is the set of the nodes of the net. An arc leads from a place p to a transition t (respectively from a transition t to a place p) if $W(p, t) > 0$ (respectively $W(t, p) > 0$). An *ordinary* net is a net whose weight function W takes its values in $\{0, 1\}$.

The *incidence matrix* of a net (P, T, W) is a place-transition matrix C such that $\forall p \in P$, $\forall t \in T$, $C[p, t] = W(t, p) - W(p, t)$, where the weight of a non-existing arc is 0. The weight function W can be represented by two place-transition matrices Pre and $Post$ defined as follows: $\forall p \in P$, $\forall t \in T$, $Pre[p, t] = W(p, t)$ and $Post[p, t] = W(t, p)$. Consequently, the incidence matrix satisfies $C = Post - Pre$.

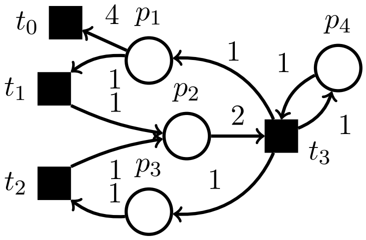
The *pre-set* of element x of $P \cup T$, denoted by ${}^\bullet x$, is the set $\{w | W(w, x) > 0\}$. By extension, for any subset E of P or T, ${}^\bullet E = \bigcup_{x \in E} {}^\bullet x$. The *post-set* of element x of $P \cup T$, denoted by $x^\bullet$, is the set $\{y | W(x, y) > 0\}$. Similarly, $E^\bullet = \bigcup_{x \in E} x^\bullet$.

We denote by max_p^N the maximum output weight of p in the net N and by gcd_p^N the greatest common divisor of all input and output weights of p in the net N. The simpler notation max_p and gcd_p is used when no confusion is possible.

A net is *homogeneous* if, for every place p, all output weights of p are equal.

A *join-transition* is a transition having at least two input places. A *choice-place* is a place having at least two output transitions.

These notions are illustrated in Figure 1.



$$C = \begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \\ p_4 \end{array} \begin{array}{c} t_0\ t_1\ t_2\ t_3 \\ \begin{bmatrix} -4 & -1 & 0 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} = \begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \\ p_4 \end{array} \begin{array}{c} t_0\ t_1\ t_2\ t_3 \\ \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array} - \begin{array}{c} \\ p_1 \\ p_2 \\ p_3 \\ p_4 \end{array} \begin{array}{c} t_0\ t_1\ t_2\ t_3 \\ \begin{bmatrix} 4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$$C \qquad\qquad Post \qquad\qquad Pre$$

Figure 1. A weighted net is pictured on the left. On the right, the incidence matrix C of the net is obtained from the matrices Pre and $Post$. The pre-set of t_3 is equal to ${}^\bullet t_3 = \{p_2, p_4\}$ and the post-set of p_1 is equal to $p_1^\bullet = \{t_0, t_1\}$. We have $max_{p_1} = 4$ and $gcd_{p_1} = gcd(1, 4) = 1$. The net is non-homogeneous since p_1 has different output weights. The transition t_3 is a join-transition since it has two inputs.

2.2. Markings, systems, firing sequences and reachability graphs

A *marking* M of a net N is a mapping $M : P \to \mathbb{N}$. A *system* is a couple (N, M_0) where N is a net and M_0 its initial marking. In this paper, notations, definitions and properties developed for nets naturally extend to systems: when applied to a system $S = (N, M)$, they concern the underlying net N of S.

A marking M of a net N *enables* a transition $t \in T$ if $\forall p \in {}^\bullet t, M(p) \geq W(p, t)$. Generalizing to sets, a set T' of transitions is enabled by M if every transition in the set T' is enabled by M. A marking M *enables* a place $p \in P$ if $M(p) \geq max_p$. Generalizing to sets, a set P' of places is enabled by M if every place in the set P' is enabled by M.

The marking M' obtained from M by firing an enabled transition t, denoted by $M \xrightarrow{t} M'$, is defined by $\forall p \in P, M'(p) = M(p) - W(p, t) + W(t, p)$.

A *firing sequence* σ on the set of transitions T is a mapping from either $\{1, \ldots, n\}$, with $n \geq 0$, or $\mathbb{N}$ to T ; it is finite of length n in the first case and infinite otherwise. When $n = 0$, σ is the empty sequence. A firing sequence $\sigma = t_1 t_2 \cdots t_n$ is *feasible* if the successive markings obtained, $M_0 \xrightarrow{t_1} M_1 \xrightarrow{t_2} M_2 \cdots \xrightarrow{t_n} M_n$, are such that for every $i \in \{1, \cdots, n\}$ M_{i-1} enables the transition t_i. We denote by $M_0 \xrightarrow{\sigma} M_n$ the fact that the firing of σ from M_0 leads to M_n.

The *Parikh vector* $\vec{\sigma} : T \to \mathbb{N}$ associated with a finite sequence of transitions σ maps every transition t of T to the number of occurrences of t in σ.

A marking M' is said to be *reachable* from the marking M if there exists a feasible firing sequence σ such that $M \xrightarrow{\sigma} M'$. The set of markings reachable from M is denoted by $[M\rangle$. The *reachability set* of a system $S = (N, M_0)$ is the set $[M_0\rangle$. The *reachability graph* of a system $S = (N, M_0)$, noted $\mathrm{RG}(S)$, is a rooted and labeled directed graph (V, E, v_0), where V is the set of markings $[M_0\rangle$, the root v_0 is the initial marking M_0 and $E = \{(M, t, M') \mid M, M' \in V \text{ and } M \xrightarrow{t} M'\}$ is the set of labeled arcs connecting every reachable marking to each of its successor markings.

Illustrations of these definitions are presented in Figure 2.

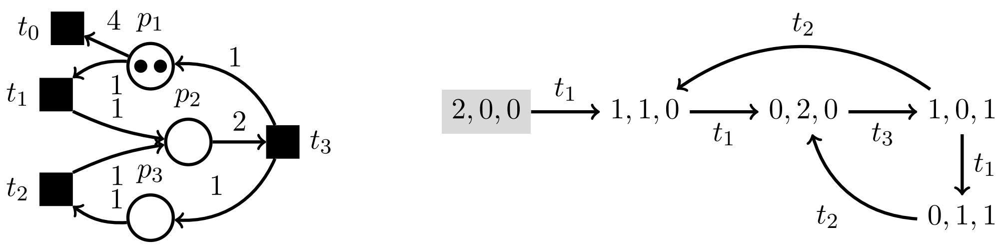


Figure 2. The weighted system pictured on the left is noted $S = (N, M_0)$ and its reachability graph is drawn on the right. The initial marking M_0, equal to $(2, 0, 0)$, is represented by the grey node in the reachability graph. The transition t_1 is initially enabled by M_0 whereas t_0, t_2 and t_3 are not. No place is enabled by M_0. The sequence $\sigma = t_1\, t_1\, t_3$, whose Parikh vector is $\vec{\sigma} = (0, 2, 0, 1)$, is feasible in S and leads to the reachable marking $(1, 0, 1)$.

2.3. Reversibility, liveness, boundedness and some related notions

A *home marking* is a marking that can be reached from any reachable marking. Formally, M is a home marking in the system (N, M_0) if $\forall M' \in [M_0\rangle, M \in [M'\rangle$. A system is *reversible* if its initial marking is a home marking.

A *deadlock* is a marking that does not enable any transition. A system (N, M_0) is *deadlock-free* if no deadlock belongs to $[M_0\rangle$.

Liveness and boundedness ensure respectively that all transitions of a system can always be fired and that the total number of tokens remains bounded. For a system $S = (N, M_0)$:

– S is *live* if for every marking M in $[M_0\rangle$ and for every transition t, there exists a marking M' in $[M\rangle$ enabling t.

– S is *bounded* if there exists an integer k such that the number of tokens in each place never exceeds k. Formally, $\exists k \in \mathbb{N}\ \ \forall M \in [M_0\rangle\ \ \forall p \in P,\ M(p) \leq k$.
S is *k-bounded* if, for any place $p \in P$, $k \geq \max\{M(p) | M \in [M_0\rangle\}$.

– S is *well-behaved* if it is live and bounded.

A marking M of a net N is live (respectively bounded) if the system (N, M) is live (respectively bounded). The structure of a net N may ensure the existence of an initial marking M_0 such that (N, M_0) is live and bounded:

– N is *structurally live* if a marking M_0 exists such that (N, M_0) is live.

– N is *structurally bounded* if the system (N, M_0) is bounded for each M_0.

– N is *well-formed* if it is structurally live and structurally bounded.

Every connected well-formed net is strongly connected [19, 32]. The algebraic properties of consistency and conservativeness are necessary conditions for well-formedness for all weighted Petri nets [32, 33]. They are defined next in terms of the existence of particular annulers of the incidence matrix.

2.4. Semiflows, consistency and conservativeness

Semiflows are particular left or right annulers of an incidence matrix C, which is supposed to be non-null:

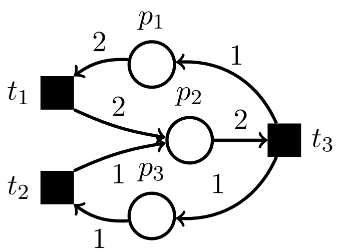
- A P-semiflow is a non-null vector $X \in \mathbb{N}^{|P|}$ such that $X^T \cdot C = \mathbb{0}$.
- A T-semiflow is a non-null vector $Y \in \mathbb{N}^{|T|}$ such that $C \cdot Y = \mathbb{0}$.

The *support* of a vector V, noted $\mathcal{S}(V)$, is the largest subset of $\mathcal{I}(V)$, the set of indices of V, associated to non-zero components of V, meaning that $\forall i \in \mathcal{S}(V)$, $V[i] \neq 0$ and $\forall i \in \mathcal{I}(V) \setminus \mathcal{S}(V)$, $V[i] = 0$. A P-semiflow is *minimal* if the greatest common divisor of its components is equal to 1 and its support is not a proper superset of the support of any other P-semiflow. Minimal T-semiflows are defined similarly.

We denote by $\mathbb{1}^n$ the column vector of size n whose components are all equal to 1. The conservativeness and consistency properties are defined as follows using the incidence matrix C of a net N:

- N is *conservative* if a P-semiflow $X \in \mathbb{N}^{|P|}$ exists for C such that $X \geq \mathbb{1}^{|P|}$, in which case X is called a *conservativeness vector*.
- N is *consistent* if a T-semiflow $Y \in \mathbb{N}^{|T|}$ exists for C such that $Y \geq \mathbb{1}^{|T|}$, in which case Y is called a *consistency vector*.

The net on Figure 3 is conservative and consistent.



$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}^T \begin{bmatrix} -2 & 0 & 1 \\ 2 & 1 & -2 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}^T$$

$$\begin{bmatrix} -2 & 0 & 1 \\ 2 & 1 & -2 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Figure 3. This weighted net is conservative (the left vector $[1, 1, 1]$ is a P-semiflow and its components are ≥ 1) and consistent (the right vector $[1, 2, 2]^T$ is a T-semiflow and its components are ≥ 1).

2.5. Choice-Free nets, Join-Free nets, subclasses and their well-formedness

The following basic subclasses of weighted Petri nets are defined by structural restrictions on the number of inputs or outputs of nodes. The study of these particular structures has contributed to enhance the understanding of the behavior of several larger classes [19, 24].

In Choice-Free nets, each place has at most one output transition, meaning that choices are not allowed. More formally, $N = (P, T, W)$ is a *Choice-Free net* if $\forall p \in P$, $|p^\bullet| \leq 1$.

In Join-Free nets, each transition has at most one input place, meaning that synchronizations are not allowed. More formally, $N = (P, T, W)$ is a *Join-Free net* if $\forall t \in T$, $|{}^\bullet t| \leq 1$.

A net N is a *Fork-Attribution* net (or FA net) if it is both Choice-Free and Join-Free. A net is a *P-net* (also known as *S-net*) if every transition has at most one input and one output. A net is a *T-net* (also known as *weighted marked graph* or *generalized event graph*) if every place has at most one input and one output.

Figure 4. On the left, a T-net. On the right, a P-net.

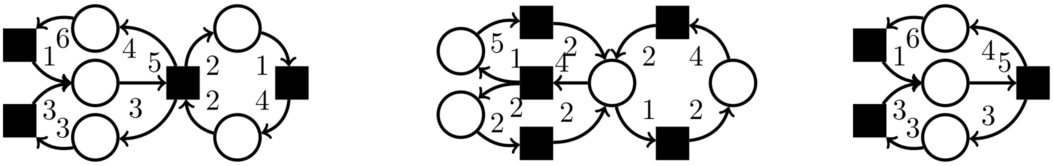


Figure 5. On the left, a Choice-Free net that is not a T-net. In the middle, a Join-Free net that is not a P-net. On the right, a Fork-Attribution net, which is both a Choice-Free and a Join-Free net.

The nets of Figures 4 and 5 capture these structural restrictions.

The following proposition expresses a necessary and sufficient condition of well-formedness for Choice-Free and Join-Free nets.

Proposition 2.1. (Well-formedness of Choice-Free and Join-Free nets [8, 24])
Suppose that N is a weighted strongly connected Choice-Free or Join-Free net. The properties

1. N is consistent and conservative
2. N is well-formed

are equivalent. Moreover, consistency implies conservativeness in strongly connected Choice-Free nets and conservativeness implies consistency in strongly connected Join-Free nets. Besides, every strongly connected well-formed Choice-Free (respectively Join-Free) net has a unique minimal T-semiflow (respectively P-semiflow) which is a consistency (respectively conservativeness) vector.

2.6. Equal-Conflict relation, sets, nets and larger classes

In order to consider nets that are more expressive than the basic Choice-Free or Join-Free classes, some choices or synchronizations must be allowed. However, in the presence of structural choices, the behavior depends on the resolution of conflicts, where the conflicts are choices whose firings disable other choices. This resolution is limited by the preconditions on the conflicting transitions and by the current marking. When these preconditions are identical, the study of the behavior is simplified. This notion of equal preconditions is captured by a relation, introduced in [19], on the transitions of any weighted net.

Let $N = (P, T, W)$ be a net. Two transitions t, t' of T are in *equal conflict relation* if $Pre[P, t] = Pre[P, t'] \neq \mathbb{0}^{|P|}$, where $Pre[P, t]$ denotes the t-th column of the matrix Pre. This is an equivalence relation on the set of transitions, and each equivalence class is an *equal conflict set*.

We deduce that an equal conflict set is enabled by a marking M if and only if at least one transition of this set is enabled by M.

A net $N = (P, T, W)$ is an *Equal-Conflict* (EC) net if for all transitions t and t' of N, ${}^\bullet t \cap {}^\bullet t' \neq \varnothing \Rightarrow Pre[P, t] = Pre[P, t']$.

A consequence of this definition is that Equal-Conflict nets are homogeneous. The Equal-Conflict class strictly extends the expressiveness of Choice-Free nets by adding the possibility of modeling choices that are equally favored. Figure 6 contains an Equal-Conflict net on the left.

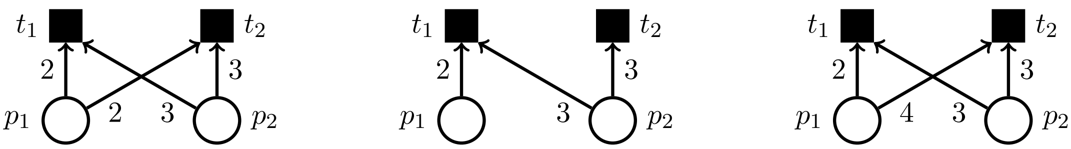


Figure 6. The net on the left is an Equal-Conflict net. In the middle, ${}^\bullet t_1 = \{p_1, p_2\} \neq \{p_2\} = {}^\bullet t_2$, hence the net is not Equal-Conflict. On the right, the pre-sets of both transitions are equal, yet the net is not Equal-Conflict since it is not homogeneous: the output weights of p_1 are not all equal.

Ordinary (unit-weighted) *Free-Choice* (OFC) nets are ordinary Equal-Conflict nets. The weighted generalization of this class encompasses the Equal-Conflict nets and is depicted on the right in Figure 6.

A net $N = (P, T, W)$, either ordinary or weighted, is *Asymmetric-Choice* (OAC or WAC) if $\forall p_1, p_2 \in P$, $p_1^\bullet \cap p_2^\bullet \neq \varnothing \Rightarrow p_1^\bullet \subseteq p_2^\bullet$ or $p_2^\bullet \subseteq p_1^\bullet$. A weighted homogeneous Asymmetric-Choice net is shown in the middle of Figure 6.

Figure 7 represents the inclusion relations between the special subclasses of weighted Petri nets considered in this paper.

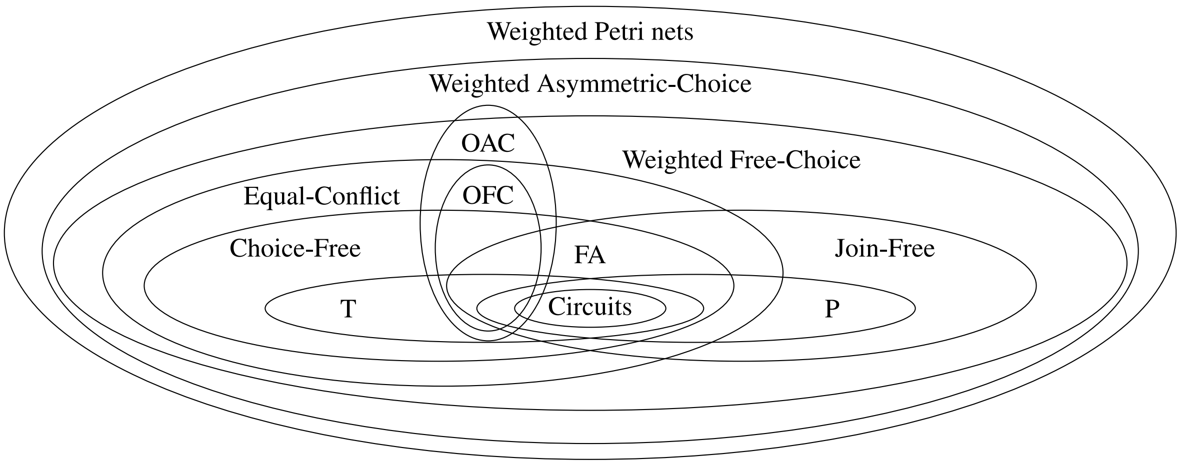


Figure 7. Some classes and subclasses of weighted systems ordered by inclusion of their structurally defined elements. A proper structural inclusion does not necessarily represent a proper expressiveness inclusion. Each class is represented by its name surrounded by an ellipse, except the Fork-Attribution (FA) class, which is represented by the intersection of the Choice-Free set with the Join-Free set.

2.7. Subsequences, subnets, subsystems, components and coverings

The sequence σ' is a *subsequence* of the sequence σ if σ' is obtained from σ by removing some of its transitions. The *projection of σ on the set $T' \subseteq T$ of transitions* is the maximum subsequence of σ

whose transitions belong to T', noted $\sigma_{|T'}$. For example, the projection of the sequence $\sigma = t_1\, t_2\, t_3\, t_2$ to the set $\{t_1,\, t_2\}$ is the sequence $t_1\, t_2\, t_2$.

Consider two nets $N = (P, T, W)$ and $N' = (P', T', W')$:

- N' is a *subnet* of N if P' is a subset of P, T' is a subset of T, and W' is the restriction of W to $(P' \times T') \cup (T' \times P')$. The system $S' = (N', M_0')$ is a *subsystem* of $S = (N, M_0)$ if N' is a subnet of N and its initial marking M_0' is the restriction of M_0 to P', *i.e.* $M_0' = M_{0|P'}$.
- N' is a *P-subnet* of N if N' is a subnet of N and $T' = {}^\bullet P' \cup P'^\bullet$. The system $S' = (N', M_0')$ is a *P-subsystem* of $S = (N, M_0)$ if N' is a P-subnet of N and S' is a subsystem of S.
- Similarly, N' is a *T-subnet* of N if N' is a subnet of N and $P' = {}^\bullet T' \cup T'^\bullet$. The system $S' = (N', M_0')$ is a *T-subsystem* of $S = (N, M_0)$ if N' is a T-subnet of N and S' is a subsystem of S.

Hence, a T-subnet (respectively P-subnet) is not necessarily a T-net (respectively P-net).

For weighted Petri nets, we define next some particular subsystems named *P- (T-)components*, which have been previously defined as subnets and studied in [19].

A (weighted) *P-component* S' of a system S is a strongly connected and conservative Join-Free P-subsystem of S. A (weighted) *T-component* S' of a system S is a strongly connected and consistent Choice-Free T-subsystem of S.

Hence, by Proposition 2.1, components are always well-formed.

We now define the union of subnets of a net and the covering of a net by some of its subnets.

Consider two nets N_1 and N_2 that are subnets of a net N. The *union* of $N_1 = (P_1, T_1, W_1)$ and $N_2 = (P_2, T_2, W_2)$ is the net $N' = (P', T', W')$ such that $P' = P_1 \cup P_2$, $T' = T_1 \cup T_2$, and the new weight function W' inherits the weights of the arcs defined by W_1 or W_2. Generalizing inductively to a set C of subnets, the union of $C = \{C_1, \ldots, C_k\}$ is the union of C_1 and the result of the union of $C \setminus \{C_1\}$. A net N is *covered* by a set C of subnets if N is the union of C.

Unions and coverings naturally extend to subsystems and systems by considering the associated restrictions of markings to subsets of places.

3. Liveness, reversibility, decompositions and T-sequences

We are interested in real applications that benefit from the combination of liveness and reversibility. Note that it follows easily from their definition that neither one of these properties implies the other in general. Nonetheless, well-behavedness implies reversibility for T-systems [6]. In this section, we study interactions of liveness with reversibility in the class of Petri nets and in several of its subclasses.

First, we provide examples of non-reversible, live, well-formed systems that belong to basic subclasses of the Equal-Conflict class. Second, we recall previous results about liveness and reversibility that exploit decompositions of well-formed Equal-Conflict systems and their subclasses. We provide a counter-example showing that one of these results, namely a known characterization of reversibility and liveness by decomposition that was developed for well-formed Choice-Free systems in [29], cannot be generalized to the well-formed Equal-Conflict class. Finally, we introduce the notion of *T-sequence* and show that the existence of a feasible T-sequence is a simple necessary condition for both liveness and reversibility in any weighted Petri net. We also recall well-known subclasses for which this condition is necessary and sufficient, before giving a counter-example for P-systems.

3.1. Non-reversibility in live, well-formed subclasses of the Equal-Conflict class

The well-formedness assumption induces strong structural and behavioral properties. Liveness implies reversibility for the class of well-formed T-systems [6] but not for several other basic classes, such as well-formed Fork-Attribution, Free-Choice and P-systems, and hence Equal-Conflict systems. These facts are illustrated in Figure 8.

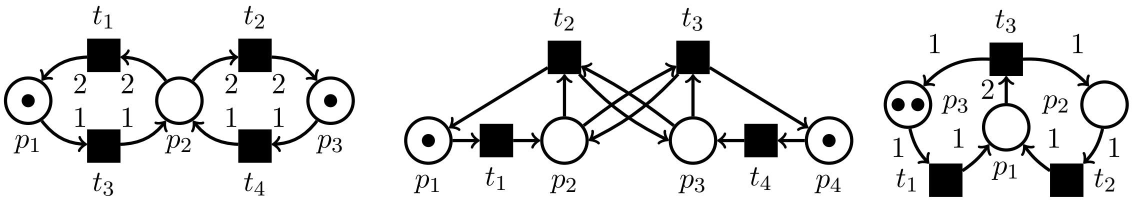


Figure 8. On the left, the homogeneous P-system is well-formed since it is live and structurally bounded: every firing preserves the total number of tokens. However, every choice in p_2 moves two tokens at once, implying non-reversibility. In the middle, the (ordinary) Free-Choice system is well-formed and live. However, giving a token back to p_1 (respectively to p_4) requires that another token be in p_3 (respectively in p_2), implying non-reversibility. On the right, the well-formed and live Fork-Attribution system is not reversible, since putting two tokens back into p_3 is not possible.

3.2. Decompositions of well-formed Equal-Conflict systems

For Equal-Conflict systems and some of their subclasses, we recall previous characterizations of liveness or reversibility that use graph decomposition methods.

For the class of ordinary Free-Choice systems, Commoner's theorem characterizes liveness in terms of particular subgraphs, *siphons* and *traps*, as equivalent to having an initially marked trap in every non-empty siphon [17, 34]. In the well-formed case, the live and reversible markings are those that mark every non-empty siphon and trap [16, 17]. However, these results cannot be readily extended to weighted nets.

In order to comprehend the behavior of Equal-Conflict systems, which subsume the ordinary Free-Choice systems, *components* have been exploited in [19, 20] instead of siphons and traps. The following proposition states that any well-formed Equal-Conflict system is covered by components, where each is well-formed and induces the support of a unique minimal semiflow according to Proposition 2.1.

Proposition 3.1. (Coverings of well-formed Equal-Conflict systems by components [20])
Consider an Equal-Conflict system S that is well-formed. S is covered both by a set of T-components, each inducing the support of a unique minimal T-semiflow, and by a set of P-components, each inducing the support of a unique minimal P-semiflow.

Thus, well-formed Equal-Conflict systems are covered by P-components, specifically conservative homogeneous Join-Free P-subsystems, which are Fork-Attribution P-components in the well-formed Choice-Free class. We recall next a characterization of liveness and reversibility for this latter case.

Proposition 3.2. (Liveness and reversibility for well-formed Choice-Free systems [29])
Consider a well-formed, hence strongly connected, Choice-Free system. It is live and reversible if and only if all its P-components are live and reversible.

This characterization does not hold in the more general case of well-formed Equal-Conflict systems, as shown in Figure 9 by the counter-example of a well-behaved non-reversible ordinary Free-Choice system whose P-components are all live and reversible.

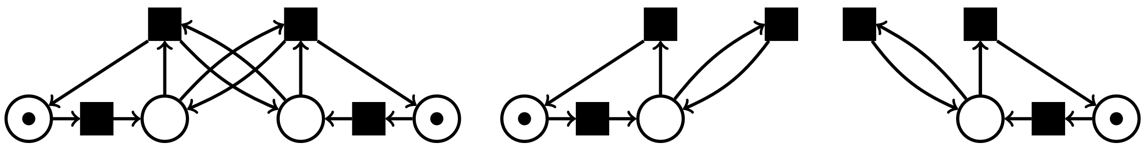


Figure 9. On the left, the well-behaved ordinary Free-Choice system is not reversible. On the right, its two P-components are live and reversible.

Since the method of decomposition into P-components cannot be applied to the Equal-Conflict class to study the reversibility property, we consider another approach in the sequel.

3.3. T-sequences

Since well-formedness and liveness are not sufficient to ensure reversibility in Equal-Conflict systems, and the decomposition into P-components does not provide insight into reversibility, we need to bring in a new assumption. For that purpose, we introduce next the notion of a *T-sequence*.

Definition 3.3. (T-sequence, partial T-sequence)
Consider a Petri net with the set of transitions T. A *T-sequence* is a sequence whose Parikh vector is equal to a T-semiflow whose support is T. A *partial T-sequence* is a sequence whose Parikh vector is equal to a T-semiflow whose support is different from T.

Hence, the Parikh vector of a feasible T-sequence is a consistency vector and a system enabling such a sequence is consistent. Alternative expressions, such as *feasible* or *realizable T-semiflow*, may be found in the literature to embody the notion of feasible (partial or not) T-sequence. Such a sequence, when feasible at the initial marking, defines *weak reversibility* in [6].

The next lemma provides a necessary condition to obtain both liveness and reversibility.

Lemma 3.4. If a system $S = (N, M_0)$ is live and reversible, then it enables a T-sequence.

Proof:
Suppose that S is live and reversible. By the liveness assumption, S enables a sequence σ_0 such that the support of $\vec{\sigma}_0$ is the set of all transitions. By the reversibility assumption, there exists a feasible sequence σ_1 returning to M_0. Thus, $\sigma_0\,\sigma_1$ is a feasible T-sequence of S. □

Consequently, any live and reversible Petri net is consistent.

In the other direction, the existence of a feasible T-sequence implies both liveness and reversibility in (weighted) Choice-Free nets [8] and (ordinary) Free-Choice nets [17] under the well-formedness assumption. However this implication is false in general. Indeed, a well-formed homogeneous P-system may have a feasible T-sequence while it is neither live nor reversible, as illustrated in Figure 10.

Moreover, a reversible system that initially enables at least one transition is deadlock-free. Since liveness is equivalent to deadlock-freeness in bounded strongly connected Equal-Conflict systems

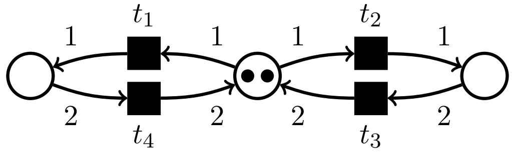


Figure 10. This well-formed homogeneous P-system is not live (fire $t_1\, t_2$) yet it enables the T-sequence $t_1\, t_1\, t_4\, t_2\, t_2\, t_3$.

[19], such a reversible system is necessarily live, which is not the case for all Equal-Conflict Petri nets.

These facts justify the study of reversibility assuming liveness in the next section, where we show that the existence of a feasible T-sequence is sufficient for reversibility in the Equal-Conflict class.

Other particular classes have been studied in [35], where the relationship between the reversibility property and the existence of reachable markings enabling a partial T-sequence associated to a minimal T-semiflow is analyzed.

4. Reversibility of live Equal-Conflict systems

In this section, we investigate the reversibility property under the liveness hypothesis for the Equal-Conflict systems, which may be unbounded and thus, not necessarily well-formed. The major result of this section is the first characterization of reversibility for live Equal-Conflict Petri nets.

First, we define a notion of fairness for sequences and develop an associated property. Then, combining the notion of T-sequence with the liveness assumption and our fairness property, we prove the first non-trivial characterization of reversibility for all live Equal-Conflict systems. This new condition states that the existence of a feasible T-sequence is sufficient for reversibility in the live Equal-Conflict class, generalizing in a weaker form conditions that have been proposed for well-formed Choice-Free and ordinary Free-Choice Petri nets. We also deduce the monotonicity of reversibility for live Equal-Conflict systems, meaning that every larger marking preserves reversibility. The liveness property was already known to be monotonic in this class and some larger ones [19, 36]. Finally, we provide some insights on the complexity of reversibility checking in this class.

4.1. Fairness in live Equal-Conflict systems

The notion of *fairness* has been exploited in numerous contexts, notably to study liveness in bounded Equal-Conflict systems [19]. We present first a slightly different definition of fairness for the Equal-Conflict class, taking inspiration from [19]. Then, we propose a result dedicated to this notion for sequences that will be used later as a tool to prove the characterization of reversibility.

Definition 4.1. (Fairness of sequences in Equal-Conflict systems)
An infinite firing sequence is *globally fair* if it fires every transition of the system an infinite number of times. An infinite firing sequence is *locally fair* if

- when an equal conflict set contains a transition that is fired an infinite number of times, all of its transitions are fired an infinite number of times, and

- when an equal conflict set is enabled, one of its transitions is fired after a finite number of firings.

We obtain the following theorem, which is a counterpart of a result of [19] for our definition of fair sequences. Comparing with [19], we replace the boundedness and strong connectedness assumptions by the liveness assumption.

Theorem 4.2. (Fairness in live Equal-Conflict systems)
Let S be a live Equal-Conflict system. An infinite sequence σ that is feasible in S is globally fair if and only if it is locally fair.

Proof:
If σ is globally fair, then the fact that σ is locally fair follows directly from the definition of fair sequences. Let us prove the converse. Suppose that σ is locally fair.

Denote by Q the set of the equal conflict sets containing at least one transition that occurs infinitely often in σ and by $\overline{Q}$ the set of the other equal conflict sets. The set Q is non-empty since there is only a finite number of equal conflict sets and σ is infinite. If $\overline{Q}$ is empty, then we are done. Now suppose that $\overline{Q}$ is non-empty.

By definition of Q and by the local fairness assumption, all transitions of the sets in Q are fired an infinite number of times in σ, while all transitions of the sets in $\overline{Q}$ are fired a finite number of times and become forever non-enabled after firing a finite prefix sequence σ_0 of σ. Denote by M the marking reached by firing σ_0 in S and by σ' the infinite suffix sequence of σ satisfying $\sigma = \sigma_0\,\sigma'$. By the liveness assumption, there exists a transition t in $\overline{Q}$ and a finite sequence σ_1 feasible at M such that σ_1 contains only transitions of Q and enables t. The sequence σ_1 may not be a prefix of σ', however all transitions of Q are fired an infinite number of times in σ'. We deduce that a finite prefix sequence σ_2 of σ' exists such that $\vec{\sigma}_2 \geq \vec{\sigma}_1$. Moreover, since only transitions of $\overline{Q}$ are structurally allowed to remove tokens from the inputs of t, the transition t becomes enabled after the firing of the finite sequence $\sigma_0\,\sigma_2$, contradicting the fact that every transition of $\overline{Q}$ stays forever non-enabled after the firing of σ_0. Thus, $\overline{Q}$ is empty and σ is globally fair. □

4.2. Sequences, orderings and delayed occurrences of transitions

We introduce notations and definitions related to particular subsequences, which will simplify our study of the reversibility property. These notions are illustrated in Figure 11.

Notations. We introduce σ^n, n being a positive integer, to denote the concatenation of the finite sequence σ taken n times, and represent its infinite concatenation by σ^∞.

The notation $K^n_{t_i}(\sigma)$, $n \geq 1$, or more simply $K^n_i(\sigma)$, denotes the largest prefix sequence of σ preceding the n-th occurrence of t_i in σ, thus containing $n-1$ occurrences of t_i. For example, considering the sequence $\sigma = t_1\,t_2\,t_1\,t_3\,t_1\,t_2\,t_3$, $K^3_{t_1}(\sigma) = t_1\,t_2\,t_1\,t_3$ and $K^1_{t_3}(\sigma) = t_1\,t_2\,t_1$.

Consider an equal-conflict set E and sequences τ, κ such that $\vec{\tau} \not\geq \vec{\kappa}$. Assume there exists a transition t in E for which $\vec{\tau}[t] < \vec{\kappa}[t]$. Consider for each transition t' in E such that $\vec{\tau}[t'] < \vec{\kappa}[t']$, its next occurrence in κ after its $\vec{\tau}[t']$-th occurrence. The transition t' in E whose next occurrence is the first to appear in κ is returned by a function, called *the next transition function* and denoted by $tnext(E, \tau, \kappa)$.

Finally, for every transition t, we denote by E^t the equal conflict set containing t.

Definitions. Consider a weighted system $S = ((P, T, W), M_0)$ and a sequence σ that is feasible in S.

For any subset of transitions $T' \subseteq T$, denote by $\sigma_{|T'}$ the subsequence of σ obtained by projection of σ on T'. The *local ordering of T' induced by σ* is the sequence $\sigma^{\infty}{}_{|T'}$.

Consider a subset $T' \subseteq T$ of transitions and a transition $t \in T'$. Denote by $\tau = \sigma^{\infty}{}_{|T'}$ the local ordering of T' induced by σ. An *occurrence* of t is *delayed* by the firing of a sequence α relatively to τ if there exists $t' \in T'$, $t' \neq t$, such that, noting $n = \vec{\alpha}[t']$ and $K = K^n_{t'}(\tau)$, we have $\vec{\alpha}[t] < \vec{K}[t]$. In other words, an occurrence of the transition t is delayed by α relatively to the local ordering τ if t occurred (strictly) fewer times in α than in the largest finite prefix sequence K of τ preceding the n-th occurrence of t' in τ. Intuitively, when t' is fired the same number of times in α as in a prefix of τ, while t is fired fewer times in α than in the same prefix, t is said to be delayed.

In the sequel, every local ordering will correspond to an equal conflict set.

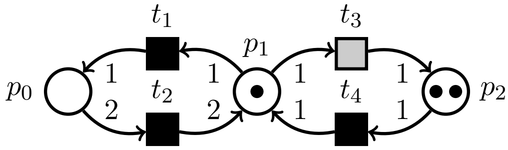


Figure 11. Consider the equal conflict sets $E^{t_1} = \{t_1, t_3\}$, $E^{t_2} = \{t_2\}$ and $E^{t_4} = \{t_4\}$, and the feasible sequence $\sigma = t_4\, t_4\, t_1\, t_3\, t_1\, t_2\, t_3$. Consider the following subsequences of σ^{∞}, which are projections of σ^{∞} on each equal conflict set and which define the associated local orderings: $\sigma_1 = (t_1\, t_3\, t_1\, t_3)^{\infty} = (t_1\, t_3)^{\infty}$, $\sigma_2 = t_2^{\infty}$ and $\sigma_4 = (t_4\, t_4)^{\infty} = t_4^{\infty}$. If the sequence $\alpha = t_3$ is fired first, then the local ordering defined by σ_1 is violated and one occurrence of t_1 is delayed: denoting $n = \vec{\alpha}[t_3] = 1$ and $K = K^n_{t_3}(\sigma_1) = K^1_{t_3}(\sigma_1) = t_1$, we have $\vec{\alpha}[t_1] = 0 < 1 = \vec{K}[t_1]$. Consequently, if one follows the local orderings and aims at removing the delay as soon as possible, the next transition to be fired in E^{t_1} is $tnext(E^{t_1}, t_3, \sigma_1) = t_1$.

4.3. A characterization of reversibility under the liveness assumption

The existence of a feasible T-sequence is a necessary condition for reversibility in all live Petri nets, as stated in Lemma 3.4. We prove that the existence of such a sequence is also a sufficient condition for reversibility in the case of live Equal-Conflict nets, thus providing the sought-for characterization of reversibility.

To conclude the proof, we shall show that, starting from any marking reached by the firing of a particular sequence σ, the T-sequence that is assumed to be feasible at the initial marking can be used as a guide to construct another sequence σ' that returns to the initial marking. Actually, as we will see later, we only need to consider all sequences σ of length 1 in order to deduce the general case. Our proof is constructive and uses two algorithms. The first one fires transitions by following local orderings until there is no more delay. The second algorithm applies to the sequence resulting from

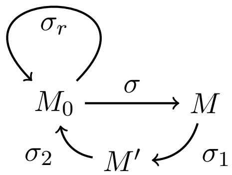


Figure 12. Consider the feasible T-sequence σ_r and the firing of σ, then Algorithm 1 builds a sequence σ_1 such that $\sigma\sigma_1$ induces no delayed occurrence, and Algorithm 2 then computes the sequence σ_2, which returns to the initial marking. The T-sequence $\sigma\sigma_1\sigma_2$ is feasible and its Parikh vector is a multiple of $\vec{\sigma}_r$.

the first algorithm and completes it to reach the initial marking. At the end, we obtain the sequence $\sigma\sigma'$ whose Parikh vector is a multiple of the Parikh vector of the initial T-sequence. These two steps are depicted in Figure 12.

4.3.1. Fairness and the firing of delayed occurrences

We assume that a T-sequence σ_r is feasible in a live Equal-Conflict system $S = (N, M_0)$. We study the Algorithm 1, which considers an initial firing of an arbitrary transition t in S, instead of an arbitrary sequence, and then fires all the delayed occurrences. If no occurrence is delayed, the empty sequence is computed instead. Algorithm 1 follows the local orderings induced by the given T-sequence σ_r in every enabled equal conflict set until all the delayed occurrences are fired. In the sequel, these local orderings are always defined on equal conflict sets. An application of Algorithm 1 is given in Figure 13.

Algorithm 1: Construction based on the feasible T-sequence σ_r of a sequence σ_t that fires the occurrences of E^t delayed by t relatively to $\sigma_r^\infty|_{E^t}$ by following the local orderings induced by σ_r in other enabled equal conflict sets.

Data: The T-sequence σ_r, which is feasible in S; the system (N, M_t) obtained by firing t in S.
Result: A sequence σ_t feasible in (N, M_t) that fires the delayed occurrences of $\kappa_0 = K_t^1(\sigma_r)$.

$\tau := t$;
while $\exists\, t' \in E^t \setminus \{t\}$, $\vec{\kappa}_0[t'] > \vec{\tau}[t']$ **do**
    while *the equal conflict set* E^t *is not enabled* **do**
        Among the transitions that belong to enabled equal conflict sets, fire the transition t_i whose next occurrence after the $\vec{\tau}[t_i]$-th appears first in σ_r^∞;
        $\tau := \tau\, t_i$;
    end
    Fire the transition $t_j = tnext(E^t, \tau, \kappa_0)$;
    $\tau := \tau\, t_j$;
end
τ is of the form $t\,\sigma_t$;
return σ_t

Figure 13. Consider the T-sequence $\sigma_r = t_1\,t_4\,t_1\,t_2\,t_3$, which is feasible in the live homogeneous P-system on the left. As first action, $t = t_3$ is fired, leading to the system on the right. Since the first output transition of p_1 to be fired in σ_r is $t_1 \neq t_3$, some occurrences are delayed, namely two occurrences of t_1. Starting from the system on the right, Algorithm 1 constructs the sequence σ_t that fires the delayed occurrences while following the local ordering in every other equal conflict set. Before the loop, $\tau = t_3$ and $\kappa_0 = K^1_{t_3}(\sigma_r) = t_1\,t_4\,t_1\,t_2$. The sequence computed is $\sigma_t = t_4\,t_1\,t_4\,t_1$, which fires the two delayed occurrences of t_1.

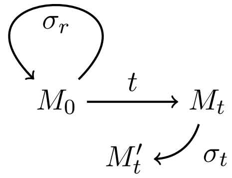


Figure 14. If the sequence σ_r is feasible, then Algorithm 1 builds the sequence σ_t feasible at M_t leading to a particular marking M_t', such that $t\,\sigma_t$ induces no delayed occurrence.

The termination and validity of this algorithm are shown next in Lemma 4.3, pictured in Figure 14. Then, Lemma 4.4 provides an equality valid at the end of the algorithm and indicates a match between occurrence counts, which will prove useful to analyze Algorithm 2.

Lemma 4.3. Let (N, M_0) be a live Equal-Conflict system in which a T-sequence σ_r is feasible. Then, for every transition t enabled by M_0, with $M_0 \xrightarrow{t} M_t$, Algorithm 1 terminates and computes the sequence σ_t that is feasible at M_t such that $t\,\sigma_t$ does not induce any delayed occurrence relatively to the local orderings based on each equal conflict set and σ_r.

Proof:
Consider the marking M_t reached by firing a transition t from M_0. We prove that Algorithm 1 computes such a sequence σ_t that is feasible at M_t.

The objective of the outer loop is to fire the transitions different from t in E^t until the number of their occurrences in τ equals that in κ_0. Every time E^t is enabled, a firing occurs in this set that follows the order of κ_0, until completion.

The objective of the inner loop is to fire transitions that do not belong to E^t by following other local orderings induced by σ_r so as to enable E^t. Let us show that the inner loop always terminates and enables E^t. First, by the liveness assumption, every reachable marking enables at least one equal conflict set. Now suppose that the inner loop does not terminate. Thus, an infinite feasible sequence τ is fired that never enables E^t. Since the firings in the loop follow the order of σ_r^∞ and the support of σ_r is T, the sequence τ is locally fair, hence globally fair by Theorem 4.2, contradicting the fact that E^t never becomes enabled. We deduce that E^t becomes enabled and the inner loop terminates.

We now prove the termination of the algorithm. Since the inner loop terminates, a transition t_j is fired at the end of each iteration of the outer loop such that $\vec{\kappa}_0[t_j] > \vec{\tau}[t_j]$ and t_j is concatenated to the current τ, reducing the number of remaining steps to attain $\vec{\kappa}_0[t_j]$. Hence the outer loop terminates. Since the stopping condition of the outer loop is the non-existence of delayed occurrences in E^t, and the firings in other equal conflict sets follow the associated local orderings, we obtain the result. □

The next lemma gives a property of the sequence $\tau = t\,\sigma_t$ obtained at the end of Algorithm 1, detailing a set of equalities matching particular occurrence counts in τ with particular occurrence counts in prefixes of κ.

Lemma 4.4. (Property of $\tau = t\,\sigma_t$)
Let $S = (N, M_0)$ be a live Equal-Conflict system in which a T-sequence σ_r is feasible. Consider the sequence σ_t constructed by Algorithm 1 after the firing of some transition t in S. Consider the sequences $\tau = t\,\sigma_t$ and $\kappa = \sigma_r^q$ where $q \geq 1$ is the smallest integer such that $\vec{\tau} \leq q \cdot \vec{\sigma}_r$. Then, $\vec{\tau}[t'] = \vec{K}_u[t']$ for each equal-conflict set E such that $t_u = tnext(E, \tau, \kappa)$ is defined and for every

transition $t' \in E$, with $m = \vec{\tau}[t_u] + 1$ and $K_u = K_u^m(\kappa)$. Besides, $\vec{\tau}[t'] = \vec{\kappa}[t']$ for every other equal conflict set E, for each transition $t' \in E$.

Proof:
Algorithm 1 terminates according to Lemma 4.3. At the end of the outer loop, for every equal-conflict set E such that $t_u = tnext(E, \tau, \kappa)$ with $\vec{\tau}[t_u] < \vec{\kappa}[t_u]$, two cases have to be considered.

If t_u does not belong to E^t, then all firings of E appeared in the same numbers and in the same order in τ as in K_u in the inner loop. We deduce that every transition t' of E satisfies $\vec{\tau}[t'] = \vec{K}_u[t']$. Otherwise, t_u belongs to E^t and the first loop fired precisely all the occurrences of E^t that belong to κ_0, in addition to the first unique firing of t. Thus, every transition t' of E^t satisfies $\vec{\tau}[t'] = \vec{K}_u[t']$. Finally, in every other equal conflict set, there is no transition t_u such that $\vec{\tau}[t_u] < \vec{\kappa}[t_u]$. Since $\vec{\tau} \le \vec{\kappa}$, we deduce the second equality. □

To illustrate, take the example of Figure 13, with $E^{t_1} = \{t_1, t_3\}$, $E^{t_2} = \{t_2\}$, $E^{t_4} = \{t_4\}$, $\tau = t_3\, \sigma_t = t_3\, t_4\, t_1\, t_4\, t_1$ and $\kappa = (\sigma_r)^2$, at the end of Algorithm 1:

For E^{t_1}, $tnext(E^{t_1}, \tau, \kappa) = t_1$, $K_1 = t_1\, t_4\, t_1\, t_2\, t_3$, $\vec{\tau}[t_1] = 2 = \vec{K}_1[t_1]$ and $\vec{\tau}[t_3] = 1 = \vec{K}_1[t_3]$.
For E^{t_2}, $tnext(E^{t_2}, \tau, \kappa) = t_2$, $K_2 = t_1\, t_4\, t_1$ and $\vec{\tau}[t_2] = 0 = \vec{K}_2[t_2]$.
For E^{t_4}, the second equality of the lemma is satisfied: $\vec{\tau}[t_4] = 2 = \vec{\kappa}[t_4]$.

4.3.2. Absence of delay and reachability of the initial marking

At the end of Algorithm 1, Algorithm 2 handles the next step of the construction. This second algorithm builds a sequence returning to the initial marking.

Algorithm 2: Computation of the feasible sequence σ'_t after the end of Algorithm 1.

Data: The sequences $\tau = t\, \sigma_t$ and $\kappa = \sigma_r^q$, where $q \ge 1$ is the smallest integer such that $\vec{\tau} \le q \cdot \vec{\sigma}_r$; the marking M'_t such that $M_0 \xrightarrow{\tau} M'_t$

Result: A completion sequence σ'_t that is feasible in (N, M'_t) such that $M'_t \xrightarrow{\sigma'_t} M_0$

```
while τ⃗ ≠ κ⃗ do
    Fire the transition t_i whose next occurrence after its τ⃗[t_i]-th appears first in κ;
    τ := τ t_i;
end
τ is of the form t σ_t σ'_t;
return σ'_t
```

Using Lemma 4.3, the following theorem shows that Algorithm 2 builds a sequence that is feasible after the firing of $\tau = t\, \sigma_t$ and reaches the initial marking. Figure 15 illustrates the theorem and Figure 16 shows an application of this second algorithm.

Theorem 4.5. Let $S = (N, M_0)$ be a live Equal-Conflict system, with $N = (P, T, W)$. Suppose there exists a feasible T-sequence σ_r in S. For every transition t enabled by M_0 such that $M_0 \xrightarrow{t} M_t$, there exists a sequence $\sigma_\triangleleft$ that is feasible at M_t such that $\sigma = t\, \sigma_\triangleleft$ is a T-sequence satisfying $\vec{\sigma} = q \cdot \vec{\sigma}_r$ for some integer $q \ge 1$.

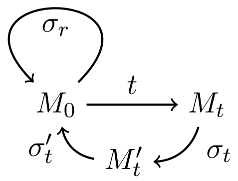


Figure 15. If the sequence σ_r is feasible and t is fired, then Algorithm 1 builds the sequence σ_t and Algorithm 2 computes the sequence σ'_t, which returns to the initial marking.

Proof:
In the rest of the proof, let $\kappa_0 = K_t^1(\sigma_r)$ be the largest prefix sequence of σ_r preceding the first occurrence of t, meaning that σ_r is of the form $\kappa_0\, t\, \sigma_2$, while the sequence κ_0 does not contain any occurrence of t. This sequence is well-defined since the support of $\vec{\sigma}_r$ is T.

If t is the first transition of E^t to be fired following the order of σ_r, meaning that κ_0 does not contain any occurrence of transitions in E^t, then the sequence κ_0 does not use any token from the input places of t. Thus, one can execute κ_0 after the firing of the first occurrence of t and the sequence $t\, \kappa_0\, \sigma_2$ is feasible at M_0. Hence, $\sigma_\triangleleft = \kappa_0\, \sigma_2$.

Otherwise, t is not the first transition in E^t to be fired following the order of σ_r, meaning that κ_0 contains at least one occurrence of another transition of E^t. We show next that Algorithm 2, whose inputs are the sequences computed by Algorithm 1, completes τ up to κ by following the order of the remaining unfired occurrences in κ. We deduce that the sequence $\sigma_\triangleleft$ obtained at the end reaches the initial marking.

To achieve this objective, we prove that the following loop invariant $I(k)$ is true for $k \geq 0$:
$I(k)$: "at the end of iteration k, for every transition t_u such that $\vec{\tau}[t_u] < \vec{\kappa}[t_u]$ and $t_u = tnext(E^{t_u}, \tau, \kappa)$, for every transition t_j of E^{t_u}, $\vec{\tau}[t_j] = \vec{K}[t_j]$, where K denotes the sequence $K_u^m(\kappa)$ and m is the value $\vec{\tau}[t_u] + 1$".

Before starting the loop, $k = 0$ and Lemma 4.4 applies.

Now assume that k iterations of the loop occurred and $I(k)$ is true. During iteration $k+1$, a new transition t_i is fired following the order of κ. At the end of iteration $k+1$, for every transition t_u such that $\vec{\tau}[t_u] < \vec{\kappa}[t_u]$ and $t_u = tnext(E^{t_u}, \tau, \kappa)$, we denote by K' the sequence $K_u^{m'}(\kappa)$ where $m' = \vec{\tau}[t_u] + 1$ and consider two cases. On the one hand, if t_u does not belong to E^{t_i}, then K' is the same sequence as in the previous iteration and for every transition t_j of E^{t_u}, $\vec{\tau}[t_j]$ has not changed either, thus $\vec{\tau}[t_j] = \vec{K'}[t_j]$. On the other hand, if t_u belongs to E^{t_i}, implying $E^{t_i} = E^{t_u}$, then K' contains the same number of occurrences of every transition t_j of E^{t_i} as in the sequence K associated to t_i in the previous iteration, except for t_i, whose number has been incremented by one. Besides, the only transition whose number of occurrences in τ has been incremented by one is t_i. Consequently, for every transition t_j of E^{t_u}, we have $\vec{\tau}[t_j] = \vec{K'}[t_j]$. We deduce finally that all the equalities that are supposed to be true at the end of iteration k remain true at the end of iteration $k+1$.

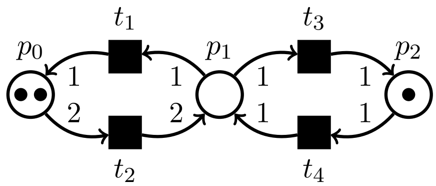
Hence, the invariant is true at every iteration of the loop. Furthermore, by definition of the t_i chosen at every step, for which we define the current value $m = \vec{\tau}[t_i] + 1$ and the sequence $K = K_i^m(\kappa)$, all occurrences in K are already present in the sequence τ of the current iteration. Thus, at the beginning of every iteration, for every transition $t_j \in T$, $\vec{\tau}[t_j] \geq \vec{K}[t_j]$.

Moreover, the sequence K is feasible at M_0 and leads to a marking that enables t_i, by definition of the feasible sequence κ. Thus, τ fired the input transitions of the input places of t_i at least as many times as in K. Then, the invariant implies that the transitions of E^{t_i} fired exactly as many times in K as in τ. Thus, the input places of t_i received at least as many tokens as they would have received by firing K from M_0, implying that t_i is enabled.

We deduce that the loop completes $\vec{\tau}$ up to $\vec{\kappa}$ and terminates.

Finally, since κ is of the form σ_r^q for some integer $q > 0$, $t\,\sigma_{\triangleleft} = t\,\sigma_t\,\sigma'_t$ is a feasible T-sequence.

□



$\kappa = (\sigma_r)^2 = t_1 \;\; t_4 \;\; t_1 \;\; t_2 \;\; t_3 \;\; t_1 \;\; t_4 \;\; t_1 \;\; t_2 \;\; t_3$

× × × × ×

$\tau = t_3\,\sigma_t = t_3 \;\; t_4 \;\; t_1 \;\; t_4 \;\; t_1$

$\sigma'_t = t_2 \;\; t_1 \;\; t_1 \;\; t_2 \;\; t_3$

Figure 16. On the left, the system obtained at the end of Algorithm 1 and the corresponding value of τ on the right. The crosses indicate the occurrences of transitions in κ that have been fired in $\tau = t\,\sigma_t$, setting $t = t_3$. In Algorithm 2, $q = 2$ and $\kappa = (\sigma_r)^2$. Following the ordering of κ, the sequence $\sigma'_t = t_2\,t_1\,t_1\,t_2\,t_3$ is fired, leading to the initial marking. Finally, after the initial firing of t_3, the sequence $\sigma_{\triangleleft} = \sigma_t\,\sigma'_t = t_4\,t_1\,t_4\,t_1\,t_2\,t_1\,t_1\,t_2\,t_3$ returns to the initial marking.

4.3.3. A characterization of reversibility

We obtain the next characterization, the proof of which is illustrated in Figure 17.

Corollary 4.6. Consider a live Equal-Conflict system $S = (N, M_0)$ such that $N = (P, T, W)$. The system S is reversible if and only if it enables a T-sequence.

Proof:

For the necessity, Lemma 3.4 applies. We prove the sufficiency next. Suppose there exists a feasible T-sequence σ_r in the live system S. We show that after the firing of any feasible sequence σ, with $M_0 \xrightarrow{\sigma} M$, there exists a feasible sequence $\sigma_{\triangleleft}$ that leads to the initial marking. For that purpose, we show by induction on the length n of σ the property $\mathcal{P}(n)$: "If a sequence σ of length n is feasible in a live Equal-Conflict system $S = (N, M_0)$ and a feasible T-sequence σ_r exists in S, then there exists a feasible sequence $\sigma_{\triangleleft}$ such that $M_0 \xrightarrow{\sigma\,\sigma_{\triangleleft}} M_0$."

If $n = 0$, σ and $\sigma_{\triangleleft}$ are empty sequences, the initial marking is reached and $\mathcal{P}(0)$ is true.

Otherwise, suppose $n > 0$, with $\sigma = t\,\sigma'$ and $M_0 \xrightarrow{t} M \xrightarrow{\sigma'} M'$, and assume that the property $\mathcal{P}(n-1)$ is true. By Theorem 4.5, there exists a sequence σ'_t that is feasible at M such that $M \xrightarrow{\sigma'_t} M_0$ and the sequence $t\,\sigma'_t$ is a T-sequence. Thus, the T-sequence $\sigma'_t\,t$ is feasible at M. Applying the induction hypothesis $\mathcal{P}(n-1)$ on the sequence σ' of size $n-1$, which is feasible in the live system (N, M), we obtain a sequence σ_d that is feasible at M' and returns to M. Thus, the sequence $\sigma_{\triangleleft} = \sigma_d\,\sigma'_t$ is feasible at M' and leads to M_0, hence $\mathcal{P}(n)$ is true. This case is illustrated in Figure 17.

We deduce that after the firing of any feasible sequence in S, there exists a feasible sequence that returns to the initial marking. We conclude that S is reversible. □

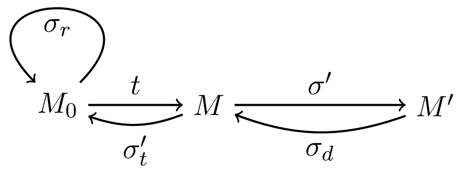


Figure 17. Suppose that the T-sequence σ_r and the sequence $\sigma = t\,\sigma'$ are feasible at M_0. Applying Theorem 4.5, the sequence σ'_t exists and leads to M_0, while $t\,\sigma'_t$ and $\sigma'_t\,t$ are T-sequences. Then, by induction hypothesis on the size of σ', σ_d exists, leading to M. Hence, the sequence $\sigma_\triangleleft = \sigma_d\,\sigma'_t$ is feasible at M' and leads to M_0.

Some examples of the previous sections provide insight into the conditions of this characterization. Indeed, Figure 8 presents non-reversible, live Equal-Conflict systems that do not enable any T-sequence. Figure 10 depicts a non-reversible, non-live Equal-Conflict system that enables a T-sequence.

4.4. Monotonicity of both reversibility and liveness in Equal-Conflict systems

We recall next the monotonicity of liveness in Equal-Conflict systems, stating that every marking larger than a live marking is also live. This result was first proved under the boundedness assumption for the Equal-Conflict class in [19], and was later generalized, without the boundedness restriction, to the more expressive DSSP class [36]. Liveness monotonicity has also been studied in [27].

Proposition 4.7. (Liveness monotonicity [19, 36])
Let (N, M_0) be an Equal-Conflict system. Consider a marking M'_0 such that $M'_0 \geq M_0$. If (N, M_0) is live then (N, M'_0) is live.

We are now able to deduce from Corollary 4.6 the following monotonicity property.

Corollary 4.8. (Reversibility and liveness monotonicity in Equal-Conflict systems)
Consider a live and reversible Equal-Conflict system $S = (N, M_0)$. Then, for any marking $M'_0 \geq M_0$, the system (N, M'_0) is also live and reversible.

Proof:
By Proposition 4.7, (N, M'_0) is also live. Moreover, Corollary 4.6 applies, meaning that S enables a T-sequence σ. The system (N, M'_0) clearly enables σ and is consequently reversible by Corollary 4.6. □

4.5. The complexity of checking the reversibility property

The general problem of checking the reversibility of a given marking in a given Petri net is decidable [11] and PSPACE-hard [12]. The formulation of our characterization of reversibility (Corollary 4.6) does not provide a checking algorithm. Moreover, constructing a feasible T-sequence has an exponential complexity, since the values of a consistency vector, hence the number of firings in a T-sequence, are exponential (in the worst case) in the size of the net. A more efficient checking algorithm would prove the existence of a feasible T-sequence without constructing it.

5. Reversibility and T-sequences in other classes: counter-examples

In the previous section, we showed that the existence of a feasible T-sequence is a necessary and sufficient condition for reversibility in all live Equal-Conflict systems, which are not necessarily bounded. However, we show next that this characterization is not true in several other classes. We provide several counter-examples, including an ordinary well-formed system (Figure 19) and an ordinary, unbounded Asymmetric-Choice system (Figure 20), all of them being strongly connected, live and not reversible while allowing a T-sequence.

Counter-example 1: An ordinary, unbounded, live system. The system on the right of Figure 18 is not reversible and is not Equal-Conflict, nor even Asymmetric-Choice. It is obtained by modifying the system on the left, which is taken from [37].

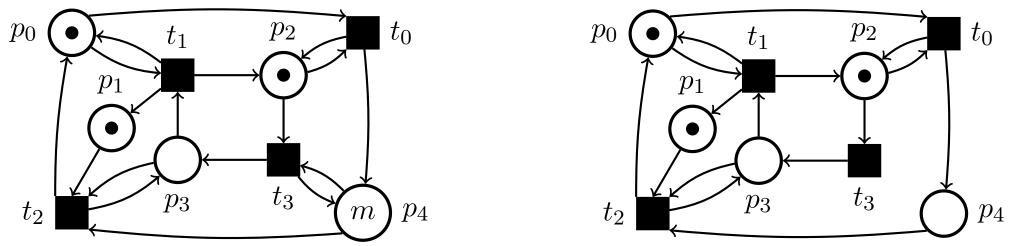


Figure 18. Consider the T-sequence $\sigma_r = t_0\, t_3\, t_2\, t_1$, which is feasible in the systems pictured for any initial value of m. On the left, if $m = M_0(p_4) = 0$, then the system is live, bounded and reversible, since every feasible sequence is a prefix of σ_r^k for some positive integer k. If $m = M_0(p_4) \geq 1$, then it is live, unbounded (fire $(t_3\, t_1)^n$ for any positive integer n) and reversible (the only unbounded place is p_1 and t_2 can be fired more often than t_1). The system on the right is live, unbounded, not reversible (t_3 can be fired without firing t_0 and t_2 cannot be fired more often than t_1). It is not Asymmetric-Choice since p_0 and p_2 are both inputs of t_0, $p_0^\bullet = \{t_0, t_1\}$, $p_2^\bullet = \{t_0, t_3\}$, thus $p_0^\bullet \not\subseteq p_2^\bullet$ and $p_2^\bullet \not\subseteq p_0^\bullet$.

Counter-example 2: An ordinary, well-formed, live system. Figure 19 illustrates a modification of the net on the right of Figure 18: the place p_1 is removed, while p_5 is added with one initial token, implying non-reversibility. If p_5 is not present, we obtain both liveness and reversibility.

Counter-example 3: An ordinary, unbounded, live Asymmetric-Choice system. We obtain in Figure 20 an ordinary Asymmetric-Choice net by deleting the arcs between p_0 and t_1 in the system on the right of Figure 18.

Counter-example 4: A well-behaved, non-conservative system. The counter-example of Figure 21[1] is well-behaved. Its reachability graph is given on the right.

Counter-example 5: A weighted, unbounded, live Free-Choice system. In Figure 22, we provide a weighted Free-Choice system that is close to Join-Free, with a single, two-input, join-transition that distinguishes it from the Join-Free class.

[1]This example, with a different layout, is taken from a Petri net course of Prof. Javier Campos, Universidad de Zaragoza, Spain.

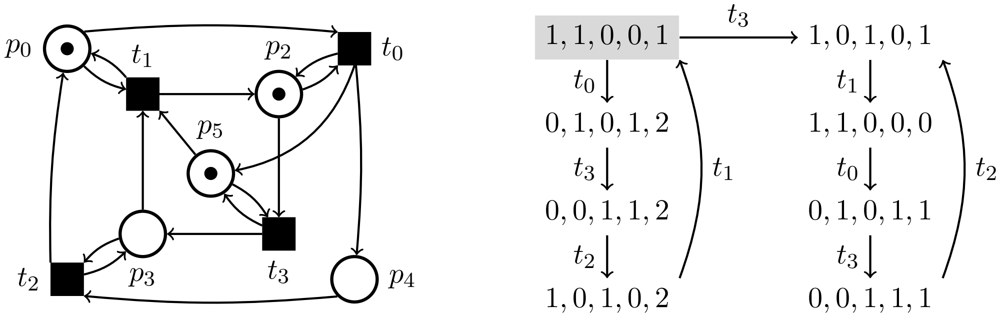
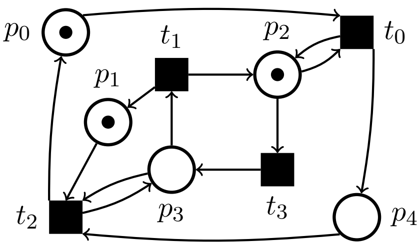
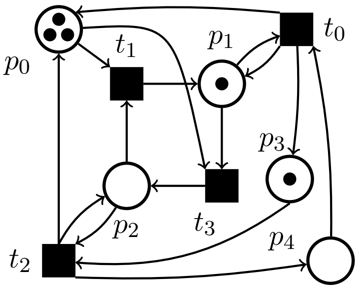


Figure 19. The system allows the T-sequence $t_0\, t_3\, t_2\, t_1$. Liveness, boundedness and non-reversibility can be deduced from its reachability graph on the right. Multiplying the input and output weights of p_0 by 2 yields a system in which any transition firing preserves the overall number of tokens. We deduce that $(2, 1, 1, 1, 1)$ is a conservativeness vector, hence the net is structurally bounded. Since it is also structurally live, it is well-formed. The system does not belong to the Asymmetric-Choice class since p_0 and p_2 are both inputs of t_0, $p_0^\bullet = \{t_0, t_1\}$, $p_2^\bullet = \{t_0, t_3\}$, thus $p_0^\bullet \not\subseteq p_2^\bullet$ and $p_2^\bullet \not\subseteq p_0^\bullet$.



Place p	Output set $p^\bullet$
p_0	t_0
p_1	t_2
p_2	t_0, t_3
p_3	t_1, t_2
p_4	t_2

Figure 20. The table on the right helps checking the Asymmetric-Choice condition for this ordinary system. It is unbounded since the place p_1 is unbounded (fire $(t_3\, t_1)^k$ for any positive integer k). It is live since t_1 and t_3 can always be fired after a finite number of firings, thus allowing new firings of t_0 and t_2. It is not reversible since there is always an occurrence of t_1 between two occurrences of t_2. The system allows the T-sequence $t_0\, t_3\, t_2\, t_1$.



$$
\begin{array}{c} \\ p_0 \\ p_1 \\ p_2 \\ p_3 \\ p_4 \end{array}
\begin{array}{c} \begin{array}{cccc} t_0 & t_1 & t_2 & t_3 \end{array} \\ \left[\begin{array}{cccc} 1 & -1 & 1 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ -1 & 0 & 1 & 0 \end{array}\right] \end{array}
$$

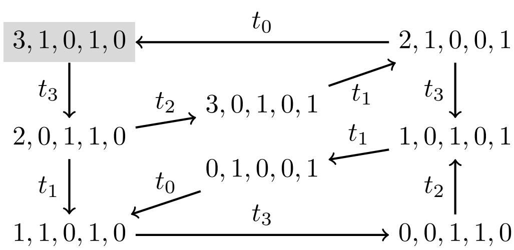


Figure 21. The Petri net on the left is well-behaved, not reversible and enables the T-sequence $t_3\, t_2\, t_1\, t_0$. Its incidence matrix shows that it is not conservative (add column t_0 to t_2), thus not well-formed. However, it is consistent, the vector $(1, 1, 1, 1)$ being the Parikh vector of the T-sequence. The reachability graph is depicted on the right, the grey state being the initial marking.

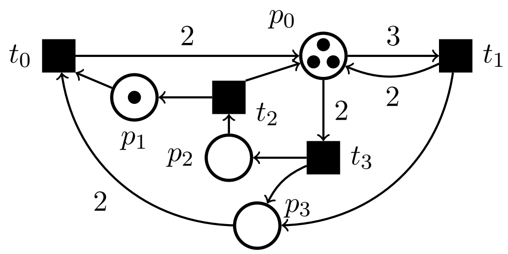


Figure 22. In this weighted Free-Choice system, the T-sequence $t_1\, t_3\, t_2\, t_0$ is initially enabled. The place p_1 is unbounded (fire the sequence $(t_3\, t_2\, t_3\, t_2\, t_0)^k$ for any positive integer k), thus the system is unbounded. Two consecutive firings of t_1 are not possible, and t_0 is either enabled by a firing of t_1 followed by a firing of t_3, or by two firings of t_3 with a firing of t_2 in between. Firing only occurrences of t_2 and t_3 generates tokens in p_1 that cannot be removed. Hence the system is not reversible. After any firing sequence, it is possible to send three tokens back to p_0 while p_1 contains one or more tokens. Such a marking enables the T-sequence, hence the system is live.

6. Polynomial sufficient condition for liveness and reversibility in well-formed Equal-Conflict systems

In this section, we focus on well-formed Equal-Conflict systems, for which we define a set of markings $\mathcal{M}_{\mathrm{JF}}$ for the homogeneous Join-Free case and a marking M_{EC} for the remaining elements of the class, namely Equal-Conflict systems that are not Join-Free. The number of tokens in these markings is linear in the size of the system and the weights.

The markings of the set $\mathcal{M}_{\mathrm{JF}}$ are already known to be live for all well-formed Join-Free systems [28, 38], which are not necessarily homogeneous. The major results of this section are the liveness of M_{EC} and the reversibility of $\mathcal{M}_{\mathrm{JF}}$ and M_{EC} in the well-formed Equal-Conflict class. By the monotonicity of liveness and reversibility in this class, we deduce the first polynomial sufficient conditions for these properties in the well-formed Equal-Conflict class. To achieve this objective, we use a previous decomposition result to ensure the liveness of M_{EC} and we show that $\mathcal{M}_{\mathrm{JF}}$ and M_{EC} enable a T-sequence in such nets, as required by Corollary 4.6 to obtain the reversibility.

The nets considered in this section are connected and well-formed, hence strongly connected [19, 32].

6.1. The polynomial markings $\mathcal{M}_{\mathrm{JF}}$ and M_{EC} for Join-Free and Equal-Conflict nets

In the following, for a net $N = (P, T, W)$, we define the particular markings $\mathcal{M}_{\mathrm{JF}}$ and M_{EC}. We also recall previous results related to these markings. First, we describe the set of markings $\mathcal{M}_{\mathrm{JF}}$, each one being identified by the choice of a place p in P.

If N is a Join-Free net, $\mathcal{M}_{\mathrm{JF}}$ is the set of markings M_p, $p \in P$, satisfying:

– $M_p(p) = max_p$,

– for every place p' in $P \setminus \{p\}$, $M_p(p') = max_{p'} - gcd_{p'}$.

The marking M_{EC} is defined as follows for the Equal-Conflict nets that contain a synchronization.

If N is an Equal-Conflict net that is not Join-Free, the marking M_{EC} satisfies:

- for all input places p of all join-transitions, $M_{\mathrm{EC}}(p) = max_p$,
- for all other places p', $M_{\mathrm{EC}}(p') = max_{p'} - gcd_{p'}$.

We recall next the monotonic liveness of the markings $\mathcal{M}_{\mathrm{JF}}$ in well-formed Join-Free nets, which are not necessarily homogeneous.

Proposition 6.1. (Monotonic liveness of the set $\mathcal{M}_{\mathrm{JF}}$ in the Join-Free case [28, 38])
Let $S = (N, M_0)$ be a well-formed, hence strongly connected, Join-Free system. S is live if M_0 is larger than or equal to a marking of $\mathcal{M}_{\mathrm{JF}}$.

In the particular case of well-formed Fork-Attribution systems, the markings of $\mathcal{M}_{\mathrm{JF}}$ are known to be live and reversible, as recalled by the next result, which was proved in [28].

Proposition 6.2. (Monotonic liveness and reversibility of $\mathcal{M}_{\mathrm{JF}}$ in the Fork-Attribution case [28])
Let $S = (N, M_0)$ be a well-formed, hence strongly connected, Fork-Attribution system. S is live and reversible if M_0 is larger than or equal to a marking of $\mathcal{M}_{\mathrm{JF}}$.

Finally, the restriction of the marking M_{EC} to the well-formed Choice-Free nets that contain at least one join-transition has been studied in [29]. We recall next its monotonic liveness and reversibility.

Proposition 6.3. (Monotonic liveness and reversibility of M_{EC} in the Choice-Free case [29])
Let $S = (N, M_0)$ be a well-formed, hence strongly connected, Choice-Free system that contains at least one join-transition. S is live and reversible if M_0 is larger than or equal to M_{EC}.

6.2. Polynomial sufficient condition of liveness for well-formed Equal-Conflict nets

We prove the monotonicity of liveness for M_{EC} in the well-formed Equal-Conflict case. For that purpose, we recall a previous decomposition result for this class and then project the marking on each relevant subsystem.

By Propositions 2.1 and 3.1, every well-formed Equal-Conflict system is covered by a set of well-formed P-components, which are strongly connected homogeneous Join-Free P-subsystems. The next proposition states the liveness of a well-formed Equal-Conflict system by observing the liveness of its P-components.

Proposition 6.4. (A characterization of liveness for well-formed Equal-Conflict systems [19])
Consider a well-formed, hence strongly connected, Equal-Conflict system S. The system S is live if and only if every (well-formed) P-component of S is live.

This characterization of liveness in terms of subsystems does not readily lead to an efficient algorithm for checking liveness, as one may have to check an exponential number of subsystems. The idea consists in making every P-component live by restriction of a particular marking, namely M_{EC}.

We prove first a simple and general technical lemma on the structure of the strongly connected Join-Free P-subnets of strongly connected nets.

Lemma 6.5. Let N be a strongly connected net with at least one join-transition. Each non-empty strongly connected Join-Free P-subnet of N contains at least one input place of a join-transition.

Proof:
Suppose there exists a strongly connected Join-Free P-subnet N_{JF} containing only transitions having a unique input place in N. Then, either N_{JF} is equal to N which would then be a Join-Free net, a contradiction, or N_{JF} is a proper subnet and there exists a node n in N_{JF} and a node n' in $N - N_{JF}$, such that n' is an input of n, since N is strongly connected. The node n cannot be a place, otherwise N_{JF} would not be a P-subnet. Hence n is a transition with at least two input places: the one in N_{JF}, N_{JF} being strongly connected, and n'. Thus N_{JF} contains a transition that is a join-transition of N, a contradiction. □

We are now able to prove the monotonic liveness of M_{EC} for the well-formed Equal-Conflict nets. The systems studied in the following theorem are not Join-Free, since this case has been considered in Proposition 6.1. In the sequel, in order to facilitate our reasonning, we say that a marking M *makes* a system $S = (N, M')$ satisfy a property $\mathcal{P}$, if the system $(N, M_{\mid P})$, with $N = (P, T, W)$, satisfies $\mathcal{P}$.

Theorem 6.6. (Polynomial live marking M_{EC})
Let $S = (N, M_{\text{EC}})$ be a well-formed, hence strongly connected, Equal-Conflict system that is not a Join-Free system. Then S is well-behaved and every larger initial marking also provides well-behavedness.

Proof:
Let $S_{JF} = (N_{JF}, M_0^{JF})$ be any of the P-components of N, where $N_{JF} = (P_{JF}, T_{JF}, W_{JF})$ and M_0^{JF} is the restriction of M_0 to P_{JF}. By definition of P-components and by Proposition 2.1, S_{JF} is a well-formed Join-Free P-subsystem of N. Moreover, by Lemma 6.5, every strongly connected Join-Free P-subnet of N contains at least an input place of a join-transition, hence contains at least a place p such that $M_0(p) = max_p$. Thus, by Proposition 6.1, S_{JF} is live. We deduce that M_0 makes every P-component live and S is well-behaved by Proposition 6.4. Moreover, by Proposition 4.7, any larger marking makes the system well-behaved. □

Figure 23 depicts a well-formed live Equal-Conflict system in which each input place p_i of a join-transition has a marking equal to max_{p_i} and all the other places p contain $max_p - gcd_p$ tokens. Figure 24 shows all the well-behaved P-components of this system.

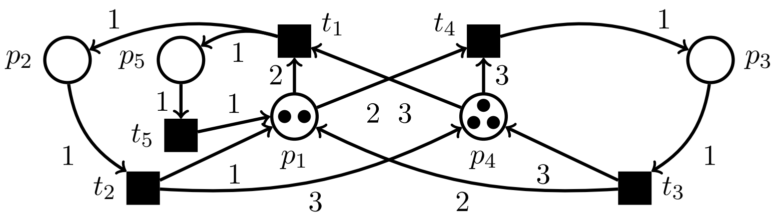


Figure 23. The Equal-Conflict system is well-formed and marked by M_{EC}, thus it is live by Theorem 6.6. Indeed, since t_1 and t_4 are the only join-transitions, their input places p_1 and p_4 are initially marked by $max_{p_1} = 2$ and $max_{p_4} = 3$ tokens. Every other place p contains $max_p - gcd_p$ tokens. Moreover, adding tokens preserves liveness.

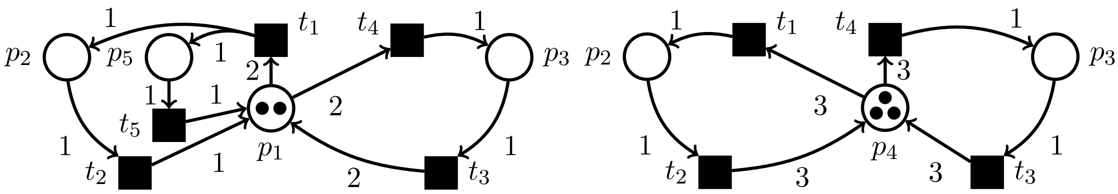


Figure 24. The well-behaved P-components of the Equal-Conflict system of Figure 23.

6.3. A simple property related to subsystems

We provide a simple general result on subsystems, which will prove useful for our further study of reversibility in this section. The lemma below, illustrated in Figure 25, brings up simple inequalities on the weights in subsystems, in which places may have fewer surrounding weights than in the complete system. As precised previously, when S is a system, the max_p^S and gcd_p^S notation applies to the underlying net of S.

Lemma 6.7. Consider any system S containing a place p with at least one input and one output. Let S' be any subsystem of S containing p and at least one of its inputs and one of its outputs. Then $max_p^S \geq max_p^{S'}$, $gcd_p^S \leq gcd_p^{S'}$, and consequently $max_p^S - gcd_p^S \geq max_p^{S'} - gcd_p^{S'} \geq 0$.

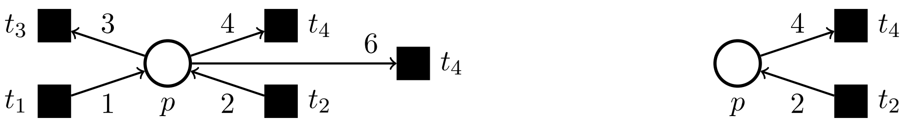


Figure 25. In the initial system S on the left, $gcd_p^S = 1$, $max_p^S = 6$ and $max_p^S - gcd_p^S = 5$. In the subsystem S' on the right, $gcd_p^{S'} = 2$, $max_p^{S'} = 4$ and $max_p^{S'} - gcd_p^{S'} = 2$.

Proof:
Since the place p may have fewer ouputs in S' than in S, $max_p^S \geq max_p^{S'}$. Since the place p may have fewer surrounding weights in S' than in S, $gcd_p^S \leq gcd_p^{S'}$ as the gcd of a non-empty subset of values can only be larger than or equal to the gcd of the complete set. We deduce trivially the last inequality. □

6.4. Enabled T-sequence in systems covered by T-components marked by $\mathcal{M}_{\text{JF}}$ or M_{EC}

In this subsection, we study a particular class of systems, any of which is denoted by the letter ζ and defined as follows.

Definition of ζ. We denote by $\zeta = (N, M_0)$ any system satisfying the following conditions:

- M_0 is greater than or equal to a marking of $\mathcal{M}_{\text{JF}}$ if N is a Join-Free net, and it is greater than or equal to M_{EC} otherwise;
- ζ is strongly connected and covered by T-components.

An example of such a system is presented in Figure 26.

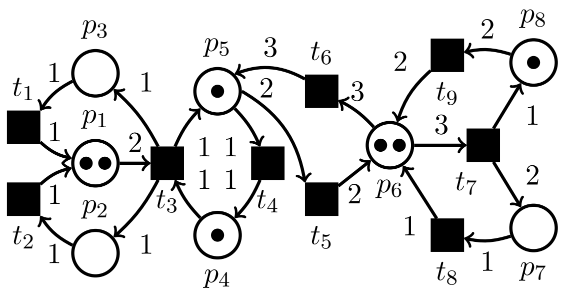


Figure 26. This system $S = (N, M_{\rm EC})$ is non-homogeneous, since p_5 has different output weights, while it is covered by three T-components, namely the Choice-Free T-component $C_1 = ((P_1, T_1, W_1), M_{01})$ that is not FA, where $T_1 = \{t_1, t_2, t_3, t_4\}$, the circuit T-component $C_2 = ((P_2, T_2, W_2), M_{02})$ where $T_2 = \{t_5, t_6\}$ and the FA T-component $C_3 = ((P_3, T_3, W_3), M_{03})$ where $T_3 = \{t_7, t_8, t_9\}$. Define the directed elementary path $\mu = p_1\, t_3\, p_5\, t_5\, p_6$. Then, the sequence $\sigma_\mu = t_3\, t_5$ is feasible at $M_{\rm EC}$.

In the following, we present several technical lemmas that describe general properties of ζ. Finally, we use these intermediate results to prove that ζ enables a T-sequence.

The next lemma describes properties of ζ related to T-components and join-transitions.

Lemma 6.8. The system ζ has the following properties:

- Every T-component of ζ that contains a join-transition is live and reversible.
- Every FA T-component C of ζ that contains a place enabled in C is live and reversible.
- For each place p enabled in ζ, all output transitions of p are enabled in ζ.

Proof:
By Lemma 6.7, every enabled place of ζ is also enabled in any T-component that contains this place. Consider a T-component C that contains a join-transition. By definition of $M_{\rm EC}$, all join-transitions and their input places are enabled in ζ, hence also in C. Every other place p of C contains at least $max_p^\zeta - gcd_p^\zeta \geq max_p^C - gcd_p^C$ tokens. Thus, by Proposition 6.3, C is live and reversible.

Consider any FA T-component C' that contains a place p' enabled in C'. Every other place p'' of C' contains at least $max_{p''}^{C'} - gcd_{p''}^{C'}$ tokens. Hence, by Proposition 6.2, C' is live and reversible.

Finally, consider any enabled place p of ζ, and any output transition t of p. If t is a join-transition, it is enabled by definition of $M_{\rm EC}$. Otherwise, t has a unique input, which is p, and t is enabled. $\square$

A *path* μ of a net N is a finite and connected sequence of nodes $\mu = n_1\, n_2 \ldots n_k$ of N, for some positive integer k. The path μ is a *directed path* if we add the constraint that, for every positive integer i, $2 \leq i \leq k$, n_i is an output of n_{i-1}. In the sequel, we only consider directed paths. A path is *elementary* (or *simple*) if it does not contain twice a same node. We denote by $\sigma_\mu = Seq(\mu)$ the firing sequence deduced from the path μ by considering its transitions sequentially. We define next particular sets of directed elementary paths of ζ that will prove useful to build a feasible T-sequence.

Definition of Γ_ζ. For any non-enabled place p of ζ, we denote by $\Gamma_\zeta(p)$ the set of all the directed elementary paths $\mu = p_0\, t_0 \ldots p$, such that p_0 and t_0 are enabled in ζ, while no other place of μ is enabled in ζ.

The following technical lemma is illustrated in Figure 26.

Lemma 6.9. For every non-enabled place p_f of ζ, the set $\Gamma_\zeta(p_f)$ is not empty. Moreover, for every (directed elementary) path $\mu = p_0\, t_0 \ldots p_f$ in $\Gamma_\zeta(p_f)$, the sequence $\sigma_\mu = Seq(\mu)$ is feasible in ζ.

Proof:
By definition of ζ and Lemma 6.8, there exists in ζ an enabled place p, whose output transitions are consequently enabled. Strong connectedness implies that there exists a directed elementary path from p to p_f, of the form $p\, t \ldots p_f$, where t is thus enabled. If this path contains another place p' enabled in ζ, another shorter path is obtained from it by removing its prefix just before p' in the path. Iterating this process leads to a satisfactory path μ. Thus, $\Gamma_\zeta(p_f)$ contains μ and is not empty.

The rest of the proof shows that $\sigma_\mu = Seq(\mu)$ is feasible.

By definition of μ, its first place p_0 is the only place of μ that is enabled by M_0. Since every input place of a join-transition is enabled by M_0, for each place $p \neq p_0$ in μ, no output of p is a join-transition.

Suppose that σ_μ is not feasible, meaning that it is of the form $\tau\, t\, \tau_1$, where τ can be fired in ζ and contains t_0, leading to a marking M that does not enable t. Denote by p_t the unique input place of t.

No output transition of p_t can be a join-transition in ζ. Then, since the path μ is elementary, p_t appears only once in μ and no output transition of p_t appears in τ. We deduce that the firing of τ did not remove any token from p_t. Moreover, p_t is an output of the transition fired last in τ, while every place p of μ is marked with at least $max_p - gcd_p$ tokens in ζ. This implies: $M(p_t) \geq M_0(p_t) + gcd_{p_t} \geq max_{p_t} - gcd_{p_t} + gcd_{p_t} = max_{p_t}$ and M enables t, a contradiction.

Consequently, σ_μ is enabled. □

The next technical lemma shows the existence in ζ of a directed path with strong properties.

Lemma 6.10. For every non-reversible or non-live T-component C of ζ, there exists a place $p \in C$ and a directed elementary path $\mu \in \Gamma_\zeta(p)$ of the form $d_1 \ldots d_u$, for some $u \geq 2$, that is covered by u different T-components $D_1, \ldots, D_u$ satisfying:

- for $i \in \{1, \ldots, u-1\}$, d_i starts with a place, ends with a transition and is the longest directed subpath of μ that is covered by D_i and that does not contain any node of $D_{i+1} \ldots D_u$,
- $d_u = p$ is the only non-empty subpath of μ that belongs to $C = D_u$,
- for every transition t of μ, if d_i is the directed subpath that contains t then there does not exist any D_j with $j > i$ that contains an input place of t.

Proof:
Since every T-component of ζ that contains a join-transition is live and reversible (Lemma 6.8), C is an FA T-component. Moreover, no place of C is enabled in C nor in ζ (Lemma 6.8).

By Lemma 6.9, for every (non-enabled) place p of C, $\Gamma_\zeta(p)$ is not empty. Hence one can select a directed elementary path $\mu = p_0\, t_0 \ldots p$ among those of $\bigcup_{p \in C} \Gamma_\zeta(p)$ that are covered by a minimal set of T-components containing C and such that p is the only node of μ that belongs to C.

We construct a new directed path μ' based on μ as follows. Consider a minimal set of T-components $Q = \{C_1, \ldots, C_u = C\}$ covering μ. First, set $\mu' = \mu$. Then, while μ' is of the form $\delta\, \delta_i\, \delta'\, \delta'_i \delta''$, where δ_i and δ'_i are non-empty directed subpaths of μ' whose nodes are covered by C_i, and δ' is a non-empty directed subpath of μ whose nodes are not covered by C_i, select a directed elementary path δ''_i in C_i starting with the first node of δ_i and ending with the last node of δ'_i. Finally, set the new μ' as $\delta\, \delta''_i\, \delta''$. Iterate these steps until all such situations have been removed.

At the end of all iterations, if μ' is not elementary, the removal of its circuits leads to a directed (hence connected) elementary subpath, from which the smallest suffix containing a place enabled in ζ is denoted by μ''. If the last place of μ'' is not the only one of μ'' that belongs to C, then simplify μ'' by removing its suffix after its first place in C. Thus, for some $p \in C$, $\mu'' \in \Gamma_\zeta(p)$. The set Q covers μ'' and contains C.

Let us define now an ordered sequence $D_1, \dots, D_u$ of the u different T-components of Q. Define D_1 as any T-component of Q that covers the longest possible prefix d_1 of μ'' such that no node of d_1 belongs to another T-component of Q. Then, for every $i \in \{2, \dots, u\}$, define D_i as any T-component of $Q \setminus \{D_1, \dots, D_{i-1}\}$ that covers the longest possible prefix d_i of μ_i such that $\mu'' = d_1 \dots d_{i-1}\mu_i$ and no node of d_i belongs to any T-component of $Q \setminus \{D_1, \dots, D_i\}$. By construction of μ'', there cannot exist two disjoint longest directed subpaths of μ'' covered by a same T-component of Q. Thus, no T-component appears twice in the sequence $D_1, \dots, D_u$.

If a T-component D_k of $\{D_2, \dots, D_u\}$ contains a join-transition or is an FA subsystem that contains a place enabled in ζ, then there is a place of D_k enabled in ζ. In this case, for some $p \in C$, there exists a directed elementary path of $\Gamma_\zeta(p)$ that is covered by the strongly connected union of $\{D_k, \dots, D_u\}$, contradicting the minimality of the covering of the initial path μ. Hence, every element of $\{D_2, \dots, D_u\}$ is an FA T-component containing no enabled place of ζ.

We prove next by induction on $n \in \{2, \dots, u\}$ that the prefix $d_1 \dots d_{n-1}$ of μ'' exists and satisfies the statement of the lemma.

If $n = 2$, suppose that d_1 is empty. Thus, the first place of μ'', which is enabled in ζ, is covered by some T-component D_j of Q, $j > 1$, contradicting the fact that no T-component of $\{D_2, \dots, D_u\}$ contains an enabled place. Hence, d_1 contains at least a place and a transition. Moreover, by definition, no place of d_1 belongs to another T-component of Q. If a transition t of d_1 has an input p_t that belongs to another T-component D_x of Q, p_t is not in d_1 and t is a join-transition. Then, p_t is enabled in ζ and belongs to D_x, a contradiction.

If $n > 2$, we suppose the claim to be true for $n-1$. Hence, there is a prefix $d = d_1 \dots d_{n-2}$ of μ'' satisfying the conditions of the lemma. Then, μ'' is of the form $d\, d_{n-1}\, d'$. If the first node of d_{n-1}, which is a place, is covered by a T-component D_y, $y > n-1$, a directed elementary path of $\Gamma_\zeta(p)$, for some $p \in C$, exists in $Q \setminus \{D_{n-1}\}$, a covering smaller than Q, a contradiction with the definition of Q. Hence, d_{n-1} is not empty. Since $D_n, \dots, D_u$ are FA T-components that cannot contain a place enabled in ζ, no transition of d_{n-1} can be a join-transition having an input place in $\{D_n, \dots, D_u\}$. By definition of a T-component, the last node of d_{n-1} is a transition. Finally, by the inductive assumption, we deduce that $d\, d_{n-1}$ does not contain any node of $\{D_n, \dots, D_u\}$ nor any transition having an input place in the latter set.

Finally, by construction of μ'', d_u also satisfies the condition of the lemma. Hence, the result. □

The following technical lemma, illustrated in Figure 27, states the possibility of making—through particular firings—any non-reversible or non-live T-component of ζ live and reversible, while retaining the reachability of the initial marking.

Lemma 6.11. For every non-reversible or non-live T-component C of ζ, there exists a sequence feasible in ζ leading to a marking M_C that makes C live and reversible. Moreover, the initial marking M_0 of ζ is reachable from M_C.

Proof:
Consider a directed elementary path $\mu = d_1 \ldots d_u$ satisfying the properties of Lemma 6.10. We prove by induction on $i \in \{1 \ldots u-1\}$ the following property $\mathcal{P}(i)$: "the marking M_i reached by firing $\sigma_i = Seq(d_1 \ldots d_i)$ makes D_{i+1} live and reversible, and M_i enables a sequence σ'_i whose transitions all belong to the components $D_1 \ldots D_i$ and which returns to M_0."

If $i = 1$, then $\sigma_i = Seq(d_1)$ fires only transitions of D_1, which is live and reversible, leading to a marking M_1 which enables a place of D_2. Since no token was removed from $D_2, \ldots D_u$ by the firing of $\sigma_i = \sigma_1$ (Lemma 6.10), the restriction of M_1 to the T-component C is greater than or equal to a marking of $\mathcal{M}_{\text{JF}}$, implying its liveness and reversibility (Proposition 6.2). By the reversibility of D_1, a sequence σ'_1 is enabled by M_1 and reaches M_0.

If $i \geq 2$, suppose that the property is true for $i-1$. Thus, noting $\mu_{i-1} = d_1 \ \ldots \ d_{i-1}$, the sequence $\sigma_{i-1} = Seq(\mu_{i-1})$, which is feasible at M_0 (Lemma 6.9), leads to a marking M_{i-1} that makes D_i live and reversible. Moreover, the sequence $\nu' = Seq(d_i)$ is feasible at M_{i-1} and leads to a marking M_i that enables the first place of d_{i+1}. No token has been removed from places in $D_{i+1}, \ldots, D_u$ by the firing of $Seq(d_1, \ldots, d_i)$ (Lemma 6.10). Thus, M_i makes D_{i+1} live and reversible (Proposition 6.2). Then, since D_i is made live and reversible by M_{i-1}, a sequence ν'' is feasible at M_i such that $\nu' \, \nu''$ is a partial T-sequence that leads to M_{i-1}. Applying the induction hypothesis, a sequence σ'_{i-1}, whose transitions all belong to the components $D_1 \ldots D_{i-1}$, is enabled by M_{i-1} and leads to M_0. Consequently, the sequence $\sigma'_i = \nu'' \, \sigma'_{i-1}$ is feasible at M_i, leads to M_0 and contains only transitions of the components $D_1 \ldots D_i$.

We deduce that the property $\mathcal{P}(i)$ is true for every $i = 1 \ldots u-1$. The sequence σ_μ leads to the marking $M_C = M_{u-1}$, which makes $D_u = C$ live and reversible, and a sequence is feasible at M_C that returns to M_0, hence the lemma. □

We now prove the enabledness of a T-sequence in ζ, as illustrated in Figure 27.

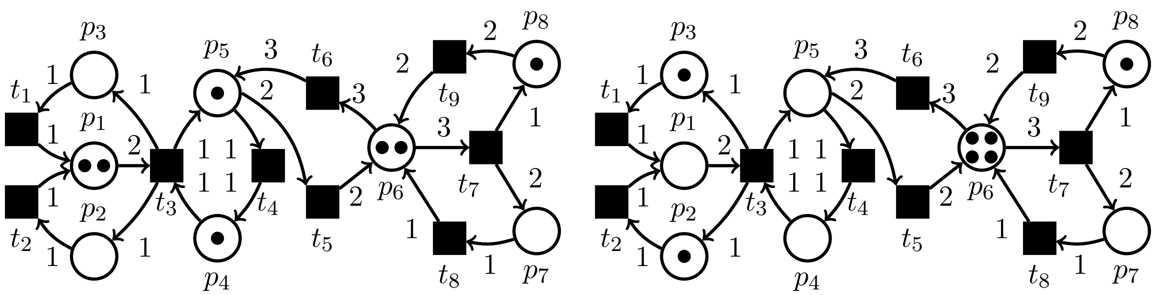


Figure 27. The system $S = (N, M_0)$ on the left is taken from Figure 26. By projecting the initial marking on the T-components C_1, C_2 and C_3, C_1 is live and reversible, while neither C_2 nor C_3 is live. Define the directed elementary path $\mu = p_1 \, t_3 \, p_5 \, t_5 \, p_6$. Hence, the sequence $\sigma_\mu = t_3 \, t_5$ is feasible at M_0, leading to the system $S_3 = (N, M_3)$ on the right which enables p_6 such that $C = ((P_3, T_3, W_3), M_{3|P_3})$ is live and reversible. Then, the partial T-sequence $\nu_3 = t_7 \, t_9 \, t_8 \, t_8 \, t_7 \, t_8 \, t_8$ is feasible in C, leading to M_3. Finally, the sequence $\sigma'_\mu = t_6 \, t_5 \, t_6 \, t_5 \, t_4 \, t_1 \, t_2$ returns to M_0. Moreover, the sequence $\sigma_\mu \, \nu_3 \, \sigma'_\mu$ is a T-sequence which is feasible in the initial system S.

Theorem 6.12. The system ζ enables a T-sequence.

Proof:
For every non-reversible or non-live T-component C of ζ, Lemma 6.11 applies and there exists a feasible sequence σ_μ that leads to a marking M_C which makes C live and reversible, while a sequence σ'_μ leads from M_C to M_0. Consequently, a partial T-sequence σ_C is feasible at M_C in ζ that contains exactly all the transitions of C. The sequence $\nu_c = \sigma_\mu\, \sigma_C\, \sigma'_\mu$ is a (partial or not) T-sequence that fires all the transitions of C.

Denote by $C_1, \ldots, C_j$ the T-components of ζ that are not both live and reversible, and by $R_1, \ldots, R_k$ the live and reversible T-components of ζ. For every $i = 1 \ldots j$, denote by γ_i a (partial or not) T-sequence that is feasible at M_0 and fires all transitions of C_i by Lemma 6.11, and, for every $i = 1 \ldots k$, denote by ρ_i a (partial or not) T-sequence that is feasible at M_0 and fires all transitions of R_i (by liveness and reversibility of R_i). Then, the sequence $\gamma_1 \ldots \gamma_j\, \rho_1 \ldots \rho_k$ is a T-sequence that is feasible in ζ. □

6.5. Polynomial sufficient conditions of liveness and reversibility

We deduce next the monotonicity of liveness and reversibility for the polynomial markings $\mathcal{M}_{\mathrm{JF}}$ and M_{EC} in the well-formed homogeneous Join-Free systems and the well-formed Equal-Conflict systems, respectively.

Theorem 6.13. (Monotonicity of liveness and reversibility for the markings $\mathcal{M}_{\mathrm{JF}}$ and M_{EC})
Let $S = (N, M_0)$ be a strongly connected well-formed Equal-Conflict system. If S is not Join-Free, suppose that M_0 is equal to or larger than the marking M_{EC}. Otherwise, if S is Join-Free, suppose that M_0 is equal to or larger than a marking of $\mathcal{M}_{\mathrm{JF}}$. Then, in both cases, S is well-behaved and reversible.

Proof:
Every well-formed Equal-Conflict net is covered by T-components (Proposition 3.1). Since S satisfies the assumptions of Theorem 6.12, it enables a T-sequence. Moreover, S is live by Theorem 6.6 and Proposition 6.1. Applying the characterization of reversibility for the live Equal-Conflict systems of Corollary 4.6, S is reversible. Morever, liveness of well-formed Equal-Conflict systems is monotonous (Proposition 4.7) and the same applies to reversibility (Theorem 4.8). □

This theorem takes advantage of the monotonicity property and thus provides a general polynomial time sufficient condition of liveness and reversibility. Hence, Figure 23 depicts a live and reversible system.

7. Conclusion

In this paper, we investigated how reversibility interacts with liveness in the Equal-Conflict Petri nets, an expressive class which allows weights, synchronizations and choices in a restricted fashion.

We introduced the notion of a T-sequence, specifically a firing sequence that visits every transition of the system and returns to the initial marking. In weighted Petri nets, the existence of a feasible T-sequence is necessary to have conjointly liveness and reversibility. For well-formed weighted Choice-Free and ordinary Free-Choice systems, constituting two proper subclasses of the Equal-Conflict class, the same condition was already known to ensure both liveness and reversibility. However, this result does not extend to well-formed homogeneous P-systems, a proper subclass of the Equal-Conflict class. By taking liveness as an assumption, we relaxed this condition and showed that the existence of a feasible T-sequence ensures reversibility in live Equal-Conflict systems, which may not be bounded nor strongly connected. Thus, we established for these systems the first non-trivial characterization of reversibility.

From this result, we deduced the monotonicity of the reversibility property for live Equal-Conflict systems, extending the monotonicity of the liveness property known for this class.

With the help of compact counter-examples, we showed that for several other live classes of weighted Petri nets the existence of a feasible T-sequence no longer ensures reversibility.

Finally, we studied the well-formed Equal-Conflict class, for which we provided the first monotonically live and reversible markings built in polynomial time with a polynomial number of tokens. By monotonicity, these markings induce a general polynomial time sufficient condition for checking liveness and reversibility in well-formed Equal-Conflict Petri nets, contrasting with the previous exponential time conditions in the literature.

Hence, liveness and reversibility, which are two major requirements of numerous artificial systems, are now better understood and benefit from new very tractable and scalable checking methods.

Other weighted subclasses of Petri nets, such as the non-homogeneous Join-Free nets and the homogeneous Asymmetric-Choice nets, may benefit from this study in the future.

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